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A simplified method of estimating noise power spectra

Kenneth M. Hanson

Los Alamos National Laboratory, MS P940
Los Alamos, New Mexico 87545 USA

ABSTRACT

A technique to estimate the radial dependence of the noise power spectrum of images is proposed in which the calculations are conducted solely in the spatial domain of the noise image. The noise power spectrum averaged over a radial spatial-frequency interval is obtained from the variance of a noise image that has been convolved with a small kernel that approximates a Laplacian operator. Recursive consolidation of the image by factors of two in each dimension yields estimates of the noise power spectrum over that full range of spatial frequencies.

Keywords: noise power spectrum, Wiener spectrum, granularity, pass-band granularity, Laplacian, binomial kernels, Gaussian pyramid, Laplacian pyramid

1. INTRODUCTION

The noise in images is recognized as an important factor in determining image quality. Noise power spectra (NPS) provide the means of characterizing image noise and play a central role in the ultimate measure of image quality, the number of noise-equivalent quanta, (NEQ).[1] This article presents a spatial-domain method of estimating the radial dependence of the noise power spectrum. The goal of this method is to provide an easy and robust measure of the NPS that can serve for day-to-day measurements of image noise. It is not intended as a replacement of Fourier methods for estimating the NPS, which remain the gold standard.

The proposed method was motivated by the granularity measure suggested in 1935 by Selwyn[2] to characterize the spatial correlations in film-grain noise. Various definitions of granularity have been used in the literature. However, they are all related to the product $G(A) = \sigma_A \sqrt{A}$, where σ_A is the rms deviation in the average over a square or circular region of area A. When the noise in the image is uncorrelated, i.e., has a flat NPS, $G(A)$ is independent of A. Variations with respect to A reflect variations of the NPS with spatial frequency.

The calculation of σ_A may be performed by taking the square root of the variance of the result of a convolution of the noise image with a kernel of constant amplitude. Thus, it is natural to consider a generalization of granularity is based on other types of kernels. Such a generalization may be useful for suppressing the high-frequency components of the uniform square kernel. For example, Hanson[3] observed that the use of the standard kernel masked the $A^{-\frac{1}{4}}$ dependence of $G(A)$ for large A that was expected for the noise in computed-tomographic images. The use of two-dimensional pyramidal kernels resolved the issue.

The proposed method uses an approach similar to the granularity calculation to quantify the radial-frequency dependence of the NPS. The calculation is based on applying to a noise image Laplacian operators, which correspond to pass-band filters. The variance of the noise image after application of the Laplacian is proportional to the NPS averaged over the mid-frequencies of the spectrum. A multiresolution Laplacian pyramid sweeps out a range of radial frequencies at which the NPS is estimated. Because the calculations for the proposed method take place only in the spatial domain, there are potentially a number of advantages over the conventional NPS.

2. NOISE POWER SPECTRUM

The noise power spectrum (NPS), also called the Wiener spectrum, is defined mathematically in terms of the Fourier transform of noise images:

$$S(f_x, f_y) = \lim_{\Delta x, \Delta y \to \infty} \frac{1}{\Delta x \Delta y} \left\langle \left| \int r(x,y)\, exp[-2\pi\imath(x f_x + y f_y)]\, dx\, dy \right|^2 \right\rangle \ , \quad (1)$$

Contact information: Email: kmh@lanl.gov; WWW: http://www.lanl.gov/home/kmh; Telephone: 505-667-1402

where $r(x, y)$ designates a noise image, which is a function of position (x, y), and the angle brackets indicate an average over an infinite ensemble of such noise images. The integration is over the area $\Delta x \Delta y$, which ideally goes to infinity. The noise power spectrum of an image is inherently a two-dimensional structure, i.e., an image in the 2D frequency domain. The units of S are $[r^2 L^2]$, namely the square of the units of r times the units of length squared. Mathematically speaking, the noise power spectrum is a useful characterization of the noise only if the noise is Gaussian distributed about a zero mean value and stationary, i.e., its covariance is independent of position in the image. Of course, in real images this condition is violated to varying degrees and one never has an infinite number of images of infinite size. The art of estimation of noise power spectra revolves around overcoming such departures from the ideal.

The typical technique for estimating the NPS for real images is based on averaging the power of the Fourier transform of noisy image samples taken from several images, or more routinely from a single noise image. It is usually important to remove the slow variations in the image intensity and to window the samples, i.e., multiply the sample by a function that tapers to zero at the edge of the image, before taking the Fourier transform. The low-frequency portion of the NPS and its interpretation tends to be difficult to handle.[4,5] The uncertainty in the estimate of S depends on the number of samples that can be included in the average since the relative rms deviation in the power in each 2D frequency bin is approximately $[N]^{-\frac{1}{2}}$ for N averaged samples. Despite the difficulties of estimating NPS, it is possible to achieve results with absolute normalization that agree between different laboratories.[6]

Since the convolution of two images in the spatial domain is equivalent to multiplication in the frequency domain of the Fourier transforms of those images, convolving a noise image with a kernel w modifies the NPS of a noise image by a factor of $|W|^2$, where W is the Fourier transform of w. Since the total power in the frequency domain is the same as in the spatial domain, the variance in the convolved image is given by

$$\sigma_w^2 = \int |W|^2 S(f_x, f_y)\, df_x\, df_y \quad . \tag{2}$$

For discretely sampled images, the range of integration for both f_x and f_x is from $-f_N$ to f_N, where $f_N = [2a]^{-1}$ is the Nyquist frequency, a being the pixel spacing, which is assumed here to be the same in both x and y directions.

From Eq. (2) we see that σ_w^2 is essentially a weighted average of the NPS over a region in Fourier space with the weight function given by the power spectrum of w. In equation form,

$$\langle S \rangle_{|W|^2} = \sigma_w^2\, a^2 K_w \quad , \tag{3}$$

where the angle brackets indicate an average, weighted by $|W|^2$, of the power spectrum of the convolution kernel w. The normalization constant K_w is the reciprocal of the total power of the kernel

$$K_w^{-1} = \int |W|^2\, df_x\, df_y = \sum_{ij} w_{ij}^2 \quad , \tag{4}$$

where the w_{ij} are the elements of the convolution kernel for discretely sampled images.

By appropriately selecting the convolution kernel, the variance in the convolved noise image, can be used to estimate the NPS for specific spatial frequencies.

3. CONVOLUTION KERNELS AND MULTIRESOLUTION PYRAMIDS

In this section, I will briefly describe binomial kernels and their use to create Gaussian pyramids. A Gaussian pyramid can be used to generate a multiscale hierarchy of Laplacian images, which effectively represent the original image processed by a bank of pass-band filters. The objective is to use the Laplacian-convolved noise images at multiple resolution scales to estimate the NPS at various radial frequencies.

3.1. Binomial Kernels

Binomial kernels provide a useful family of convolution kernels for smoothing images. The hierarchy begins with the definition of a 2×2 kernel

$$\mathcal{B}^1 = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad , \tag{5}$$

which is normalized so the sum over all elements is unity. Subsequent orders of binomial kernels are obtained by convolution, so, in general,

$$\mathcal{B}^{m+n} = \mathcal{B}^m \circledast \mathcal{B}^n \quad , \tag{6}$$

where $\circledast$ represents convolution. The size of $\mathcal{B}^n$ is $(n+1) \times (n+1)$. These kernels all have unit sum. They are reasonable approximations to Gaussian kernels, particularly as n increases. As described by Jähne,[7] the binomial kernels, or filters as he calls them, can be used to construct many types of smoothing kernels with characteristics in the frequency domain that can be tailored to one's needs. For this work, I will use only the simple binomial filters.

An important property of the binomial kernels is that they are separable into an outer product of a row vector times a column vector. This property translates into calculational efficiency because the convolution of $\mathcal{B}^n$ with an image can be accomplished by convolving a $(n+1) \times 1$ kernel followed by a convolution of a $1 \times (n+1)$ kernel, which involves less computation than required convolution by an arbitrary nonseparable $(n+1) \times (n+1)$ kernel. The computational savings grows as n increases.

From the above binomial kernels, one can construct Laplacian operators. The lowest order Laplacian is

$$\mathcal{L}^2 = \mathcal{I} - \mathcal{B}^2 = \begin{bmatrix} -0.0625 & -0.1250 & -0.0625 \\ -0.1250 & 0.7500 & -0.1250 \\ -0.0625 & -0.1250 & -0.0625 \end{bmatrix} \quad , \tag{7}$$

where $\mathcal{I}$ is essentially a delta-function, in this case, a 3×3 kernel with unity in the central position and zeros elsewhere. The next higher-order Laplacian would be

$$\mathcal{L}^4 = \mathcal{B}^2_{\bullet} - \mathcal{B}^4_{\bullet} = \begin{bmatrix} -0.0039 & -0.0156 & -0.0234 & -0.0156 & -0.0039 \\ -0.0156 & 0.0000 & 0.0312 & 0.0000 & -0.0156 \\ -0.0234 & 0.0312 & 0.1094 & 0.0312 & -0.0234 \\ -0.0156 & 0.0000 & 0.0312 & 0.0000 & -0.0156 \\ -0.0039 & -0.0156 & -0.0234 & -0.0156 & -0.0039 \end{bmatrix} \quad , \tag{8}$$

where the dots indicate that the central elements of the kernels involved should be aligned, which is required for clarity because $\mathcal{B}^2$ is 3×3 and $\mathcal{B}^4$ is 5×5.

The 2D power spectrum of $\mathcal{L}^4$ is shown in Fig. 1. The mean radial frequency (weighted by $f df$ to obtain the proper radial mean) is 0.540 f_N along the frequency axes and 0.583 f_N along the diagonals, where F_N is the Nyquist frequency. From the 2D spectrum, it can be seen that $\mathcal{L}^4$ samples the spatial frequency domain in a nearly isotropic manner. The mean radial frequency over all angles is 0.559 f_N. The width of the radial profile is about 0.38 f_N FWHM and the normalization for Eq. (3) is $K = 49.8$. The 2D power spectrum of $\mathcal{L}^2$ is decidedly nonisotropic because it extends right up to the corners of the 2D Fourier domain delineated by $\pm f_N$. The mean radial frequency is 0.812 f_N along the frequency axes and 1.088 f_N along the diagonals, with a mean over all angles of 0.917 f_N. The normalization is $K = 1.561$. A useful feature of the Laplacian convolution is that it effectively rejects the contributions from frequencies that are much lower than the mid frequencies of the corresponding filter.

One approach to proceeding toward lower frequencies would be to recursively convolve the image with broader and broader binomial kernels and compute the variance of the Laplacians formed by taking differences between the images of different resolution so formed. This calculation is not as inefficient as it might seem at first because of the separability of the binomial kernels. One characteristic of this approach is that the noise image stays the same size.

Alternatively, similar results can be achieved with a multiresolution representation in which the smoothed images at each resolution scale are consolidated, or subsampled, to produce a progression of smaller and smaller images that can be convolved with fixed-size kernels.

3.2. Gaussian Pyramid

The idea of repeatedly consolidating an image to obtain successively smaller images of coarser and coarser resolution is captured by the Gaussian pyramid scheme.[7,8] This concept also is the basis of multiscale, multigrid, and multiresolution methods and filter banks. The zeroth level of the Gaussian pyramid is the initial full-resolution image, which is designated as G^0. Subsequent degrees of lower resolution are obtained by smoothing the image with a Gaussian

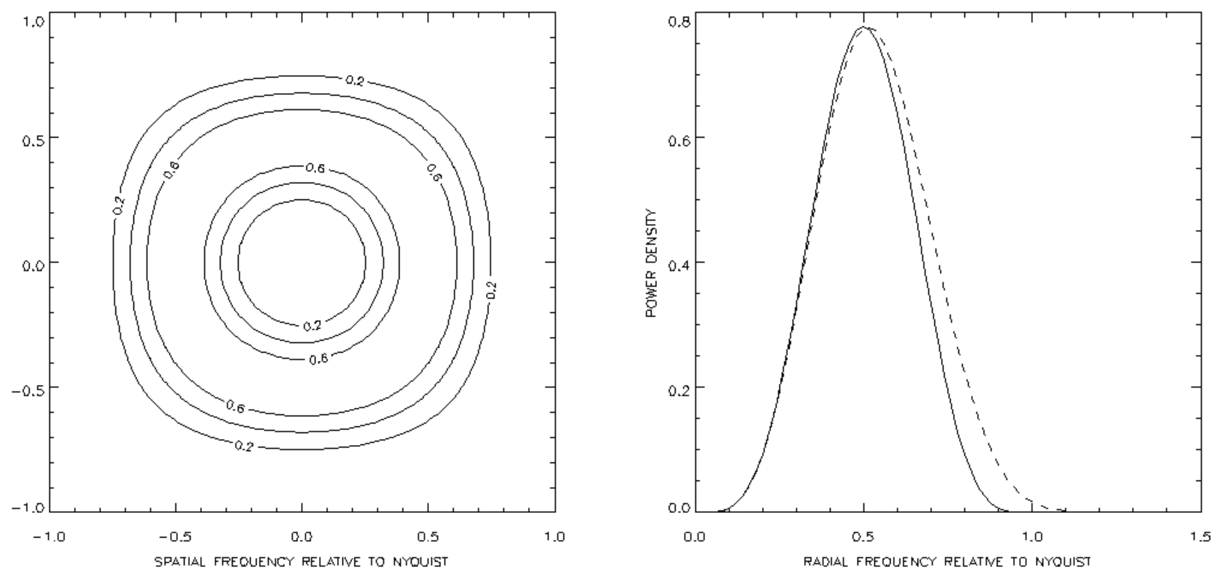


Figure 1. The two dimensional frequency contour plot of the square of the Fourier transform of $\mathcal{L}^4$ (left), renormalized to make its total power unity, and its radial frequency dependence (right) along the axis (solid line) and along the diagonal (dashed). Zero frequency of the 2D frequency plot on the left is in middle of the image.

kernel $\mathcal{G}$ and decimating, or consolidating, the result by a factor of two by taking every second pixel and every second row, a process represented by $\mathcal{R}$. The image at the kth level of the Gaussian pyramid is, therefore,

$$G^{k+1} = \mathcal{R}\mathcal{G}G^k \quad . \tag{9}$$

If the image G^k possesses components at frequencies above the Nyquist frequency for the new pixel spacing, those components will appear in G^{k+1} at lower frequencies because of the consolidation process, a phenomenon known as aliasing. For noise power spectra, the result of this aliasing is to add to the true power spectrum below the Nyquist frequency contributions from above. The essential trick is to consolidate the noise image only after convolving it with a sufficiently broad kernel that there remains little or no noise power above the new Nyquist frequency.

A Laplacian image at the kth level can be formed as follows:

$$L^k = G^k - \mathcal{G}G^k \quad . \tag{10}$$

A computational advantage of this form is that the second term can be used to obtain G^{k+1} by (9). The hierarchy of Laplacian images formed in this way is called a Laplacian pyramid.

I will use this Laplacian pyramid to estimate NPS at frequencies below one half the Nyquist frequency of the original image. For $\mathcal{G}$, I propose the fourth-order binomial kernel $\mathcal{B}^4$ discussed above. This kernel corresponds to a filter that drops nearly to zero by one half the Nyquist frequency of the image on which it operates. Thus, the consolidation operation results in little aliasing. Jähne suggests the conservative approach of using $\mathcal{B}^8$ to construct Gaussian pyramids to avoid aliasing effects. However, this approach may not be needed because we are dealing here with the power spectrum, not the Fourier amplitudes and the fraction of aliasing contributions should be reduced by their square.

Figure 2 shows the power spectrum of the effective filter at the first level ($k = 1$) for the Laplacian (10). This power spectrum is the product of the power spectrum for $\mathcal{I} - \mathcal{B}^4$ times that for $\mathcal{B}^4$ at the zeroth level. The variance of the noise image obtained at this point in the Gaussian pyramid has contributions from radial frequencies with a mean value of 0.67 $f_{N,1}$ (the Nyquist frequency for the pixel spacing at this level) with relatively small contributions from above $f_{N,1}$ caused by aliasing. The normalization constant for Eq. (3) for this kernel is $K = 6.24$, including a

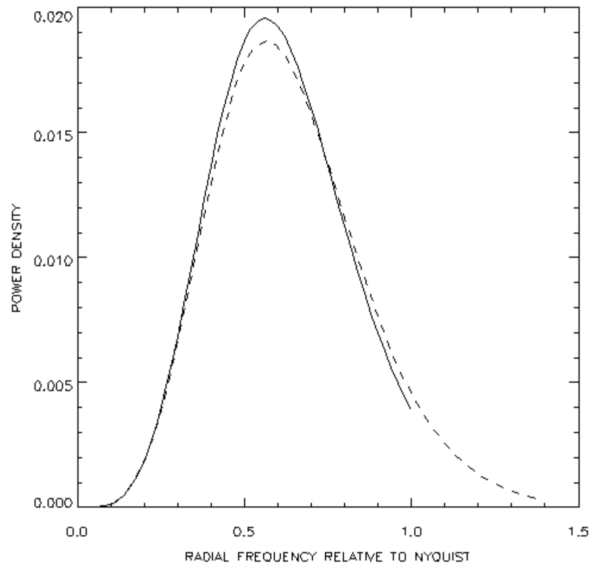


Figure 2. The radial frequency dependence of the power spectrum of the Laplacian at the first level of the Gaussian pyramid along the frequency axis (solid line) and along the diagonal (dashed) of the 2D spectrum. The Nyquist frequency at this level is one-half that of the original image.

5% reduction to account for aliasing. As one proceeds down the pyramid, K is slightly affected by the $\mathcal{B}^4$ smoothing from higher levels. It needs to multiplied by a factor of $1.37^{0.23(k-1)}$ for each level k.

3.3. Algorithm

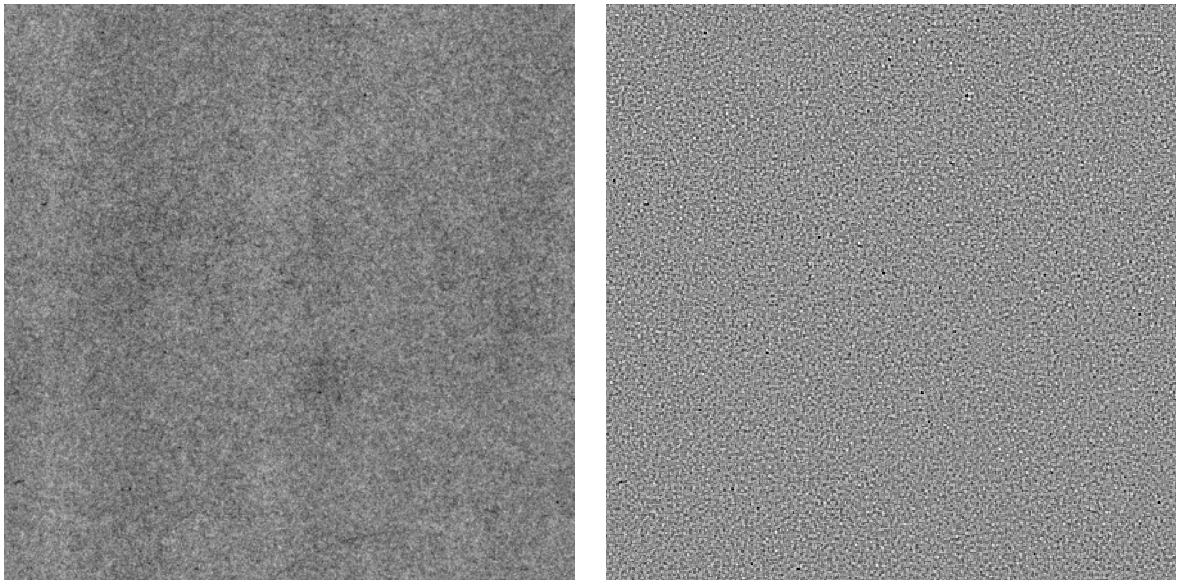
This algorithm is based on calculating the variance of a noise image after convolution with Laplacians. For the NPS estimates in the two highest-frequency intervals, the two Laplacian operators (7) and (8) are used. Estimates for lower-frequency intervals are obtained using the Laplacian pyramid.

Designating the original image as G^0, the proposed algorithm is as follows:
(1) subtract from G^0 the result of its convolution with $\mathcal{B}^2$ to obtain $\mathcal{L}^2 G^0$;
the variance of this result times $1.56\,a^2$ is S at radial frequency of 0.917/2a
(2) subtract from $\mathcal{B}^2 G^0$ the result of its convolution with $\mathcal{B}^2$ to obtain $\mathcal{L}^4 G^0$;
the variance of the result times $49.8\,a^2$ is S at radial frequency of $0.559/2a$
The image G^0 is the beginning of the Gaussian pyramid. Begin loop, starting with $k = 0$ ($a_0 = a$):
(3) consolidate $\mathcal{B}^4 G^k$ by factor of two putting every other pixel and row into G^{k+1} and increment k;
(new pixel spacing is twice previous one , i.e., $a_{k+1} = 2a_k$)
(4) subtract from G^k the result of its convolution with $\mathcal{B}^4$;
variance of the result times $6.24\,a_k^2\,\Pi_k(1.37)^{0.23(k-1)}$ is S at radial frequency of $0.67/2a_k$
Repetition of steps (3) and (4) many times generates estimates of S at geometrically decreasing frequencies.

The standard deviation in these estimates is approximately $c\,[2n_p]^{-\frac{1}{2}}$, where n_p is the number of pixels in the convolved image that can be used to calculate the variance and c is scaling constant that depends on the convolution kernel; $c = \sqrt{|w \circledast w|^2/|w|^2}$.

4. EXAMPLE

The noise image was taken from a storage-phosphor screen (Fuji BAS 3), 20 cm × 40 cm in size, that was scanned with a Fuji model BAS 1500 at the normal pitch of 100 μm. The screen was exposed to a bremstrahlung beam produced by a microtron running at 15 MeV after being filtered through 4 cm of depleted uranium. The dose on the



a b

Figure 3. (a) A 768×768 pixel^2 noise image obtained from a storage-phosphor screen exposed to high-energy x rays and (b) the Laplacian of Eq. (10) at the first level of the Gaussian pyramid. The actual linear dimension of (b) in pixels is half that of (a), but it is displayed here with the same size to emphasize the pass-band nature of the Laplacian pyramid.

screen is estimated to have been about 400 mR (4 mGy), Co^{60} equivalent, yielding scanner readings of about half of the saturation level. The full image is flattened by dividing by a fit to the original image that is cubic in x and y. Thus the analyzed noise image is dimensionless, its noise fluctuation values are relative to the mean amplitude of the original image. A 7.68-cm-square subsample of the full-size image is used in the present analysis, which is shown in Fig. 3.

The NPS estimates obtained using the algorithm outlined above are presented in Fig. 4. The error bars shown indicate the estimated rms uncertainties. The lowest-frequency point comes from a 24×24 pixel^2 image after consolidation of the original 768^2 noise image by a factor of $2^5 = 32$.

For comparison, also shown is the NPS obtained using a standard Fourier-transform technique. This calculation commences by removing the very low frequencies from the same 768^2 noise image by zeroing out the first three frequency components of the Fourier transform of the image, i.e., eliminating spatial frequencies equal to and below 0.04 mm^{-1}. Eleven overlapping 128×128-pixel^2 subsamples are extracted from the full image in each direction. Each subsample is multiplied by a 2D Hanning window and Fourier transformed. The final NPS is the average of all 121 Fourier power spectra, normalized to account for the effect of the Hanning window. The match between the Fourier results shown in Fig. 4 and those of the proposed algorithm is very good, except for the point at 0.106 mm^{-1}. The strong disagreement there is most likely caused by the fact in the Fourier analysis, the spatial frequency components below 0.04 mm^{-1} are removed.

The spatial-domain calculation for this image takes 45 s on a PC with an AMD K6 processor running at 166 MHz, whereas the Fourier transform method takes about five times longer.

Figure 5 shows the full two-dimensional NPS for the sample noise image obtained with the Fourier-transform technique. It is readily seen that this NPS is not isotropic. In particular, there are sharp peaks in the spectrum at mid frequencies. These peaks have been observed to be reproducible from image to image and are essentially eliminated when difference images between successive scans are analyzed. These observations indicate that these anomalies are probably produced by the scanner. These results show how the 2D NPS may be useful to diagnose an imaging system, as was also demonstrated in Ref. 3.

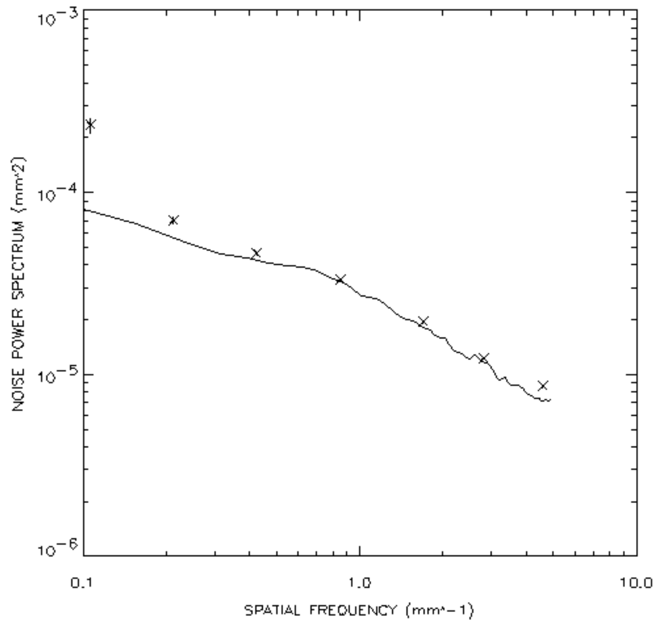


Figure 4. Noise power spectrum for the noise image shown in Fig. 3 as a function of radial spatial frequency estimated using the proposed spatial-domain technique (crosses) compared to that obtained using a conventional Fourier transform technique (line).

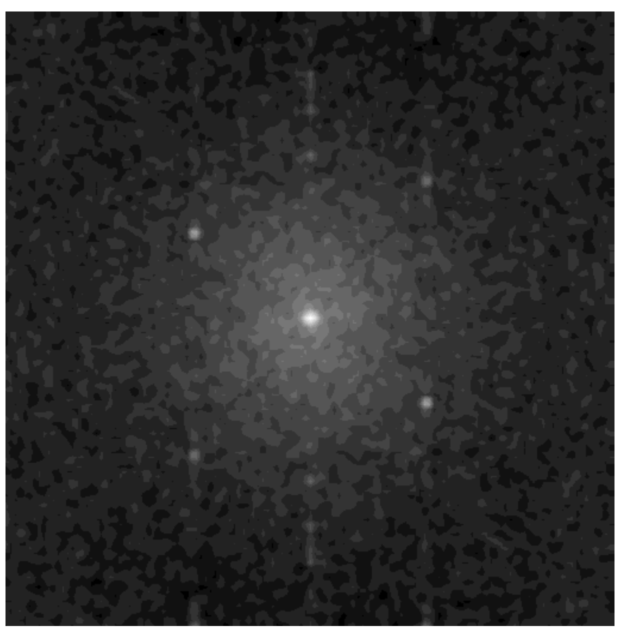


Figure 5. The two-dimensional noise power spectrum obtained with the Fourier-transform method displayed on a logarithmic intensity scale. Zero frequency is in middle of this 2D frequency-domain image.

5. DISCUSSION

I have proposed a spatial-domain method to estimate the radial dependence of the noise power spectrum from a noise image. The method is based on the convolution of a noise image with Laplacian kernels, used in conjunction with a multiresolution analysis afforded by a Gaussian pyramid. Properly scaled, the variances in the convolved images provide estimates of the noise power spectrum at radial spatial frequencies that differ by roughly powers of two.

Compared with Fourier methods, the proposed algorithm is simple to understand and reasonably easy to implement. This algorithm has a statistical efficiency comparable to that of the Fourier tranform method. Its disadvantages are poorer frequency resolution and lack of a 2D frequency display of the NPS, which can be helpful in understanding an imaging system.[3]

Because the calculations are done in the spatial domain, the proposed algorithm can be easily adapted to visually (and quantitatively) check whether the noise in the analyzed image is stationary. For example, it would be easy to display the local variance at each resolution level. By directly observing the Gaussian or Laplacian images, as in Fig. 3b, one can view the spatial correlations of noise, which may vary with position, intensity, etc., and perhaps capture the spirit of Ref. 9. With this algorithm, one can remove the effects of identified artifacts and nonstationary behavior from the NPS estimates.

The $\mathcal{B}^4$ binomial kernel used to construct the Gaussian pyramid clearly has its limitations. Jähne[7] describes a variety of smoothing kernels that could be used to improve the frequency resolution of the NPS estimates and to better suppress spurious low-frequency contributions estimates, as well as reduce aliasing effects. Other extensions of the present technique would be to use other types of kernels to restrict the NPS to selected directions, which might be important for images that are digitzed by asymmetric scanners, for instance.

The spatial-domain algorithm presented here is available on the web at `http://bayes.lanl.gov/~kmh/nps` as an IDL[10] procedure.

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