# Probability depends on status of information

⇒ Absolute probability makes no sense
(even in coin tossing!)

⇒ only conditional probability

$$P(E/I)$$

If I changes then also P changes:

- In many cases: "irrationally"
  ↳ ≡ "not easy to model"

- In most of scientific problems:
  updated through Bayes' Theorem

$$P(E|I) = \frac{P(E \cap I)}{P(I)}$$

39)
39)



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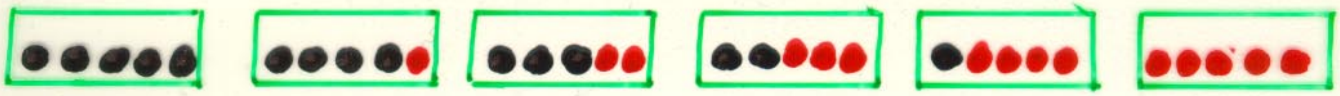
# Six boxes problem



$H_0$ $H_1$ $H_2$ $H_3$ $H_4$ $H_5$

- Symmetric status of preparation
- Choose one box:

  → Which one is it? $H_0, H_1, \dots, H_5$?

  → If I extract one ball, what will come out? ↗ Red : $E_1$? ↘ Black : $E_2$?

- - I am uncertain on $H_i$, but $\boxed{\bigcup_{i=0}^{5} H_i = \Omega}$

  certainty

  - I am uncertain on $E_i$, but $\boxed{E_1 \cup E_2 = \Omega}$

  - In general, I am uncertain about the "constituents" ("which box and which outcome?)

$E_1 \cap H_0$, $E_1 \cap H_1$, $E_1 \cap H_2$, . . . . , $E_1 \cap H_5$

$E_2 \cap H_0$, $E_2 \cap H_1$, $E_2 \cap \bar{H}_2$, , $E_2 \cap H_5$

Since both $E_i$ and $H_j$ form complete classes of hypotheses $\begin{pmatrix} \cup_j H_j = \Omega \\ \cup_i E_i = \Omega \end{pmatrix}$

$\Rightarrow E_i = \cup_j (E_i \cap H_j)$ "$E_i$ whatever $H_j$ might be"

$\Rightarrow H_j = \cup_i (E_i \cap H_j)$

$$\Rightarrow \begin{cases} P(E_i) = \sum_j P(E_i \cap H_j) = \sum_j P(E_i \mid H_j) \cdot P(H_j) \\ P(H_j) = \sum_i P(E_i \cap H_j) = \sum_i P(H_j \mid E_i) \cdot P(E_i) \end{cases}$$

Easiest thing to evaluate?

$\rightarrow P(E_i \mid H_j)$

$E_i$ plays the role of observable: effects

$H_j$ plays the role of physical hypotheses ("true values") $\rightarrow$ cause of the effects

$\Rightarrow$ we see effects $\rightarrow$ guess causes ("infer")



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Inference: $P(H_j \mid \text{data})$

Poincaré:

Now, these problems are classified as *probability of causes*, and are the most interesting of all from their scientific applications. I play at *écarté* with a gentleman whom I know to be perfectly honest. What is the chance that he turns up the king? It is $\frac{1}{8}$. This is a problem of the probability of effects. I play with a gentleman whom I do not know. He has dealt ten times, and he has turned the king up six times. What is the chance that he is a sharper? This is a problem in the probability of causes. It may be said that it is the essential problem of the experimental method.

→ after an effect has been observed, we tend to believe to some causes more than to others

→ repeating the experiment the beliefs about the different causes are constantly updated

⇒ how?

→ $P(H_j \mid E^{(1)})$

$P(H_j \mid E^{(1)} \cap E^{(2)})$

$\vdots$

$P(H_j \mid \underbrace{E^{(1)} \cap E^{(2)} \cap \ldots\ldots \cap E^{(n)}}_{\text{entire set of observations}})$



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$$P(E_i \mid H_j)$$

Red — Black

$P(E_1 \mid H_0) = 0$ — $P(E_2 \mid H_0) = 1$

$P(E_1 \mid H_1) = 1/5$ — $P(E_2 \mid H_1) = 4/5$

$\cdots$ — $\cdots$

$\boxed{P(E_1 \mid H_j) = j/5 \qquad P(E_2 \mid H_j) = \frac{5-j}{5}}$

$\cdots$ — $\cdots$

$P(E_1 \mid H_5) = 1$ — $P(E_2 \mid H_5) = 0$

Before first observation : what shall we observe?

→ $P(E_1)$ ?

$\frac{1}{2}$ by symmetry

→ let's evaluate it in a general way valid also when the problem might (will!) become asymmetric

$$P(E_1) = \sum_j P(E_1 \cap H_j) = \sum_j \underbrace{P(H_j)}_{\to\ 1/6\ \text{because initial symmetry}} \cdot P(E_1 \mid H_j)$$

$$P(E_1 \mid \text{initial symmetry}) = \underbrace{\frac{1}{6}}_{P_0(H_j)} \times \left( \frac{0+1+2+3+4+5}{5} \right)$$

Let us now make the experiment

Outcome $\rightarrow$ $\boxed{\text{Black}}$ $\leftarrow$

$(E_2)$

What can we say now about $H_j$

in the light of the observation $E^{(1)} = E_2$?

(We don't really need to wait: the conclusions can be evaluate in advance <u>under the hypothesis</u> that $E^{(1)}$ will be observed)

$$\underbrace{P(H_j \mid E_2)}_{\text{"after" the observation}} = \underbrace{\frac{P(E_2 \mid H_j)}{P(E_2)}}_{\text{"before" the observation}} \qquad P(E_2) \neq 0$$

- "under the condition $E_2$"
- "under the hypothesis that $E_2$ is true"

Rewriting the r.h.s. of the previous formula, making appearing the "easy things":

$$\sum_j \quad P(H_j|E_i) = \frac{P(E_i \cap H_j)}{P(E_i)} = \frac{P(E_i|H_j)\cdot P(H_j)}{\sum_j P(E_i|H_j)\cdot P(H_j)}$$

Bayes' Theorem

Written in this way, we don't need any more the denominator (just a normalization factor)

$$P(H_j|E_i) \propto P(E_i|H_j)\cdot P_0(H_j)$$

"after" "posterior" "final" | "likelihood" | "before" "prior" "initial"

$P(H_j \mid \text{Information})$

| | $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $P(E_1)$ |
|---|---|---|---|---|---|---|---|
| Initial | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 0.5 |
| $E_1$ | 0 | 0.07 | 0.13 | 0.20 | 0.27 | 0.33 | 0.73 |
| $E_1$ | 0 | 0.02 | 0.07 | 0.16 | 0.29 | 0.45 | 0.82 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $E_1$ | | | | | | | 0.73 |
| $E_2$ | 0 | 0.20 | 0.30 | 0.30 | 0.20 | 0 | 0.5 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $E_2$ | 0.33 | 0.27 | 0.20 | 0.13 | 0.07 | 0 | 0.27 |
| $E_2$ | 0.45 | 0.29 | 0.16 | 0.07 | 0.02 | 0 | 0.18 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $E_2$ | | | | | | | 0.27 |
| $E_1$ | 0 | 0.20 | 0.30 | 0.30 | 0.20 | 0 | 0.5 |

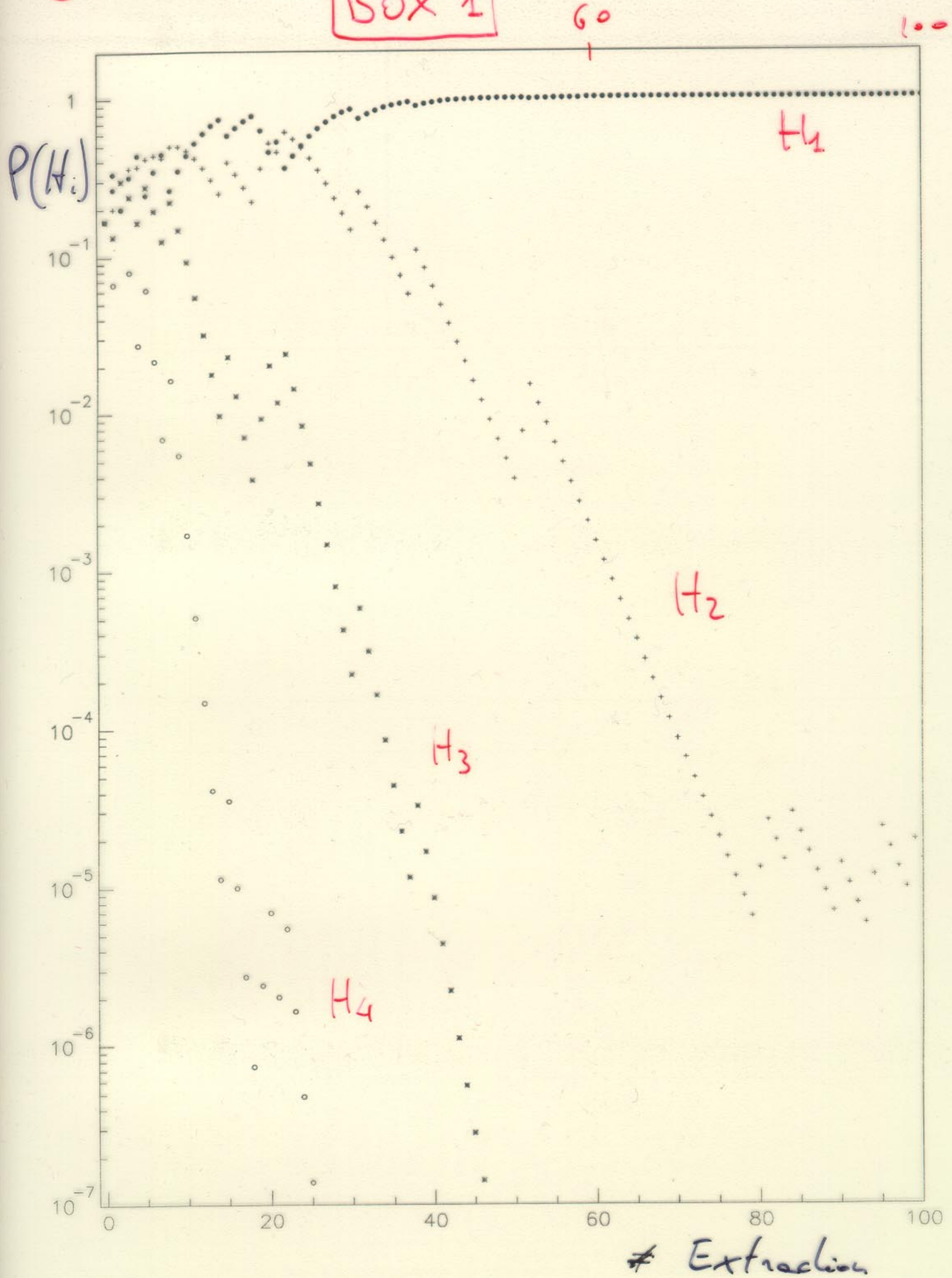
$$P(H_j \mid E_1, E_2) = P(H_j \mid E_2, E_1)$$

A scatola nr. 1 69
pw = 0.200

| | | | $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | ( 0, 1) | 0.3333 | 0.2667 | 0.2000 | 0.1333 | 0.0667 | $\boxed{0.0000}$ |
| 2 | 1 | ( 1, 1) | $\boxed{0.0000}$ | 0.2000 | 0.3000 | 0.3000 | 0.2000 | 0.0000 |
| 3 | 0 | ( 1, 2) | 0.0000 | 0.3200 | 0.3600 | 0.2400 | 0.0800 | 0.0000 |
| 4 | 0 | ( 1, 3) | 0.0000 | 0.4384 | 0.3699 | 0.1644 | 0.0274 | 0.0000 |
| 5 | 1 | ( 2, 3) | 0.0000 | 0.2462 | 0.4154 | 0.2769 | 0.0615 | 0.0000 |
| 6 | 0 | ( 2, 4) | 0.0000 | 0.3459 | 0.4378 | 0.1946 | 0.0216 | 0.0000 |
| 7 | 0 | ( 2, 5) | 0.0000 | 0.4452 | 0.4226 | 0.1252 | 0.0070 | 0.0000 |
| 8 | 1 | ( 3, 5) | 0.0000 | 0.2628 | 0.4990 | 0.2218 | 0.0164 | 0.0000 |
| 9 | 0 | ( 3, 6) | 0.0000 | 0.3495 | 0.4976 | 0.1474 | 0.0055 | 0.0000 |
| 10 | 0 | ( 3, 7) | 0.0000 | 0.4381 | 0.4678 | 0.0924 | 0.0017 | 0.0000 |
| 11 | 0 | ( 3, 8) | 0.0000 | 0.5243 | 0.4199 | 0.0553 | 0.0005 | 0.0000 |
| 12 | 0 | ( 3, 9) | 0.0000 | 0.6047 | 0.3632 | 0.0319 | 0.0001 | 0.0000 |
| 13 | 0 | ( 3,10) | 0.0000 | 0.6771 | 0.3050 | 0.0179 | 0.0000 | 0.0000 |
| 14 | 0 | ( 3,11) | 0.0000 | 0.7401 | 0.2501 | 0.0098 | 0.0000 | 0.0000 |
| 15 | 1 | ( 4,11) | 0.0000 | 0.5830 | 0.3939 | 0.0231 | 0.0000 | 0.0000 |
| 16 | 0 | ( 4,12) | 0.0000 | 0.6550 | 0.3320 | 0.0130 | 0.0000 | 0.0000 |
| 17 | 0 | ( 4,13) | 0.0000 | 0.7194 | 0.2735 | 0.0071 | 0.0000 | 0.0000 |
| 18 | 0 | ( 4,14) | 0.0000 | 0.7752 | 0.2210 | 0.0038 | 0.0000 | 0.0000 |
| 19 | 1 | ( 5,14) | 0.0000 | 0.6309 | 0.3597 | 0.0094 | 0.0000 | 0.0000 |
| 20 | 1 | ( 6,14) | 0.0000 | 0.4577 | 0.5219 | 0.0204 | 0.0000 | 0.0000 |
| 21 | 0 | ( 6,15) | 0.0000 | 0.5326 | 0.4555 | 0.0118 | 0.0000 | 0.0000 |
| 22 | 1 | ( 7,15) | 0.0000 | 0.3601 | 0.6159 | 0.0240 | 0.0000 | 0.0000 |
| 23 | 0 | ( 7,16) | 0.0000 | 0.4317 | 0.5539 | 0.0144 | 0.0000 | 0.0000 |
| 24 | 0 | ( 7,17) | 0.0000 | 0.5053 | 0.4862 | 0.0084 | 0.0000 | 0.0000 |
| 25 | 0 | ( 7,18) | 0.0000 | 0.5780 | 0.4171 | 0.0048 | 0.0000 | 0.0000 |
| 26 | 0 | ( 7,19) | 0.0000 | 0.6471 | 0.3502 | 0.0027 | 0.0000 | 0.0000 |
| 27 | 0 | ( 7,20) | 0.0000 | 0.7102 | 0.2883 | 0.0015 | 0.0000 | 0.0000 |
| 28 | 0 | ( 7,21) | 0.0000 | 0.7660 | 0.2332 | 0.0008 | 0.0000 | 0.0000 |
| 29 | 0 | ( 7,22) | 0.0000 | 0.8138 | 0.1858 | 0.0004 | 0.0000 | 0.0000 |
| 30 | 0 | ( 7,23) | 0.0000 | 0.8536 | 0.1462 | 0.0002 | 0.0000 | 0.0000 |
| 31 | 1 | ( 8,23) | 0.0000 | 0.7445 | 0.2550 | 0.0006 | 0.0000 | 0.0000 |
| 32 | 0 | ( 8,24) | 0.0000 | 0.7954 | 0.2043 | 0.0003 | 0.0000 | 0.0000 |
| 33 | 0 | ( 8,25) | 0.0000 | 0.8383 | 0.1615 | 0.0002 | 0.0000 | 0.0000 |
| 34 | 0 | ( 8,26) | 0.0000 | 0.8737 | 0.1262 | 0.0001 | 0.0000 | 0.0000 |
| 35 | 0 | ( 8,27) | 0.0000 | 0.9022 | 0.0978 | 0.0000 | 0.0000 | 0.0000 |
| 36 | 0 | ( 8,28) | 0.0000 | 0.9248 | 0.0752 | 0.0000 | 0.0000 | 0.0000 |
| 37 | 0 | ( 8,29) | 0.0000 | 0.9425 | 0.0575 | 0.0000 | 0.0000 | 0.0000 |
| 38 | 1 | ( 9,29) | 0.0000 | 0.8913 | 0.1087 | 0.0000 | 0.0000 | 0.0000 |
| 39 | 0 | ( 9,30) | 0.0000 | 0.9162 | 0.0838 | 0.0000 | 0.0000 | 0.0000 |
| 40 | 0 | ( 9,31) | 0.0000 | 0.9358 | 0.0642 | 0.0000 | 0.0000 | 0.0000 |
| 41 | 0 | ( 9,32) | 0.0000 | 0.9511 | 0.0489 | 0.0000 | 0.0000 | 0.0000 |
| 42 | 0 | ( 9,33) | 0.0000 | 0.9629 | 0.0371 | 0.0000 | 0.0000 | 0.0000 |
| 43 | 0 | ( 9,34) | 0.0000 | 0.9719 | 0.0281 | 0.0000 | 0.0000 | 0.0000 |
| 44 | 0 | ( 9,35) | 0.0000 | 0.9788 | 0.0212 | 0.0000 | 0.0000 | 0.0000 |
| 45 | 0 | ( 9,36) | 0.0000 | 0.9840 | 0.0160 | 0.0000 | 0.0000 | 0.0000 |
| 46 | 0 | ( 9,37) | 0.0000 | 0.9879 | 0.0121 | 0.0000 | 0.0000 | 0.0000 |
| 47 | 0 | ( 9,38) | 0.0000 | 0.9909 | 0.0091 | 0.0000 | 0.0000 | 0.0000 |
| 48 | 0 | ( 9,39) | 0.0000 | 0.9932 | 0.0068 | 0.0000 | 0.0000 | 0.0000 |
| 49 | 0 | ( 9,40) | 0.0000 | 0.9949 | 0.0051 | 0.0000 | 0.0000 | 0.0000 |
| 50 | 0 | ( 9,41) | 0.0000 | 0.9962 | 0.0038 | 0.0000 | 0.0000 | 0.0000 |
| 51 | 1 | (10,41) | 0.0000 | 0.9923 | 0.0077 | 0.0000 | 0.0000 | 0.0000 |
| 52 | 1 | (11,41) | 0.0000 | 0.9848 | 0.0152 | 0.0000 | 0.0000 | 0.0000 |
| 53 | 0 | (11,42) | 0.0000 | 0.9885 | 0.0115 | 0.0000 | 0.0000 | 0.0000 |
| 54 | 0 | (11,43) | 0.0000 | 0.9914 | 0.0086 | 0.0000 | 0.0000 | 0.0000 |
| 55 | 0 | (11,44) | 0.0000 | 0.9935 | 0.0065 | 0.0000 | 0.0000 | 0.0000 |
| 56 | 0 | (11,45) | 0.0000 | 0.9951 | 0.0049 | 0.0000 | 0.0000 | 0.0000 |
| 57 | 0 | (11,46) | 0.0000 | 0.9963 | 0.0037 | 0.0000 | 0.0000 | 0.0000 |
| 58 | 0 | (11,47) | 0.0000 | 0.9973 | 0.0027 | 0.0000 | 0.0000 | 0.0000 |
| 59 | 0 | (11,48) | 0.0000 | 0.9979 | 0.0021 | 0.0000 | 0.0000 | 0.0000 |
| 60 | 0 | (11,49) | 0.0000 | 0.9985 | 0.0015 | 0.0000 | 0.0000 | 0.0000 |
| 61 | 0 | (11,50) | 0.0000 | 0.9988 | 0.0012 | 0.0000 | 0.0000 | 0.0000 |
| 62 | 0 | (11,51) | 0.0000 | 0.9991 | 0.0009 | 0.0000 | 0.0000 | 0.0000 |
| 63 | 0 | (11,52) | 0.0000 | 0.9993 | 0.0007 | 0.0000 | 0.0000 | 0.0000 |
| 64 | 0 | (11,53) | 0.0000 | 0.9995 | 0.0005 | 0.0000 | 0.0000 | 0.0000 |
| 65 | 0 | (11,54) | 0.0000 | 0.9996 | 0.0004 | 0.0000 | 0.0000 | 0.0000 |

(A)

BOX 1



60
100
1
$10^{-1}$
$10^{-2}$
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
P(H_i)
H_1
H_2
H_3
H_4
0
20
40
60
80
100
# Extraction

Let us consider simulation (A)
with $n=50$ ( 9 ~~White~~ Red + 41 Black )

$\Rightarrow$ which $H_i$ ?

Consider binomial process with

$$p_i \equiv P(W \mid H_i) = \frac{i}{5}$$

$X$ : "# White in $n$ extractions"

$n=50$ , $x=9$

$$P(H_i \mid x, n) = \frac{f(x \mid \mathcal{B}_{n,p_i}) \cdot P_0(H_i)}{\sum_j f(x \mid \mathcal{B}_{n,p_j}) \cdot P_0(H_j)}$$

$$P(H_1 \mid x=9, n=50) = \frac{\frac{50!}{9!\,41!}\left(\frac{1}{5}\right)^9\left(\frac{4}{5}\right)^{41} \cdot \frac{1}{6}}{\left( \sum_j \cdots\cdots \right)}$$

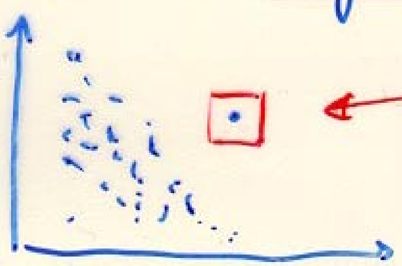
$$\sum_j = \frac{1}{6}\frac{50!}{9!\,41!}\left[\left(\frac{0}{5}\right)^9\left(\frac{5}{5}\right)^{41} + \left(\frac{1}{5}\right)^9\left(\frac{4}{5}\right)^{41} + \left(\frac{2}{5}\right)^9\left(\frac{3}{5}\right)^{41} + \cdots\cdots + \left(\frac{5}{5}\right)^9\left(\frac{0}{5}\right)^{41}\right]$$

$$\boxed{\rightarrow 0.996}$$

⇒ The conclusions of the Bayesian inference depend only on $\begin{cases} \text{- observations} \\ \text{- previous knowledge} \end{cases}$

⇒ They don't depend on the way how data are treated (if done consistently with Bayesian updating procedure)

⇒ In particular, they don't depend on what the experimentalist wanted to observe before taken data



conventional approach: the whindow must have been choosen before

⇒ Nice paper

J. O. Berger & D. A. Berry, "≈ Illusion of objectivity" Am. Scientist, 76 ('88) 159

Similar problem (forget for the moment the previous!)

- A box could contains 0, 1, 2, 3, 4, 5 ~~white~~ Red balls out of 5 total balls.

- What is the probability of $H_0, \ldots, H_5$?

- Extract and reintroduce balls, as in the previous problem:

$$\rightarrow P(H_i \,|\, E_n^{(n)}) \;?$$

Same solution? It could be, but it could as well not be!

The 5 balls could have been chosen randomly from a large amount of balls, of which 50% ~~white~~ red and 50% black

$$P_0(H_i) = (H_i \,|\, E_0) = \frac{5!}{i!(5-i)!}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{5-i}$$

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P_0(H_i)$ | 0.031 | 0.156 | 0.313 | 0.313 | 0.156 | 0.031 |

83)

$P(H_j \mid Info)$

| | $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | |
|---|---|---|---|---|---|---|---|
| Initial | 0.03 | 0.16 | 0.31 | 0.31 | 0.16 | 0.03 | 0.5 |
| $E_1$ | 0 | 0.06 | 0.25 | 0.38 | 0.25 | 0.06 | 0.60 |
| $E_2$ | 0 | 0.02 | 0.16 | 0.38 | 0.33 | 0.10 | 0.67 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $E_1$ | | | | | | | 0.60 |
| $E_2$ | 0 | 0.13 | 0.38 | 0.38 | 0.13 | 0 | 0.5 |

⇒ Differences on conclusions on $P(H_i | E_n)$?

⇒ What is the right solution?

⇒ It makes no sense to speak about the "right solution":

→ Priors depend on the status of knowledge (subjective)

→ after many "extractions" the conclusions will depend very little from priors

→ "practically" objective conclusions

Poincaré (Science et Hypothèse)

An effect may be produced by the cause *a* or by the cause *b*. The effect has just been observed. We ask the probability that it is due to the cause *a*. This is an *à posteriori* probability of cause. But I could not calculate it, if a convention more or less justified did not tell me in advance what is the *à priori* probability for the cause *a* to come into play—

"On en recherche constamment de pareilles en physique :
Les lois ne nous sont connues
que par leurs effets qu'on observe.
Chercher à en déduire les lois qui
sont des causes, c'est résoudre un
problème de probabilité des causes".

— Calcul des probabilités

Let's consider again

$$P\left(E^{(2)} = E_1 \mid E^{(1)} = E_2\right) = 0.27$$

It was calculated from

$$P\left(E_1 \mid E^{(1)} = E_2\right) = \sum_j \underbrace{P(E_1 \mid H_j)} \cdot P\left(H_j \mid E^{(1)} = E_2\right)$$

in principle
$P(E_1 \mid H_j, E^{(1)} = E_2)$
but independent..

$$= 0 \times 0.33 + \frac{1}{5} \times 0.27 + \frac{2}{5} \times 0.2 + \frac{3}{5} \times 0.13 +$$

$$+ \frac{4}{5} \times 0.067 + 1 \times 0 = \underline{\underline{0.267}}$$

<u>What means</u>? "we believe 27% that $E^{(2)}$ will be Red, given the available information"

(already said...) ⇒

## Note

1) 27 % has no interpretation in terms of favorable/possible events (rather easy to understand)

2) 27% has no interpretation in terms of long run frequency either (and this need some discussion...)

⇒ Thinking of measuring $P(E_1 \mid E^{(1)}, E_2)$ makes no sense!

If we imagine to make an experiment to measure the frequency for "$n \to \infty$" we shall get, <u>most likely</u> 0%, or 20%, or 40%, or 60%, 80%, <u>but not</u> 27%

⇒ "The Bayesian conclusion is clearly wrong" ?

But <u>who said that 27% means frequencies?</u>

⇒ 27% is simply the <u>rational belief</u> that the <u>next ball will be red</u>, based on the best use of all available information in hand

<u>nothing more!</u>

Does it mean that if the experiment is made and $\simeq 20\%$ frequency is obtained (with $n = 100$ trials), a Bayesian will defend his position?

YES, in the sense that for him $P(E^{(2)} = E_1 \mid E^{(1)} = E_2)$ <u>was</u> 27%!
(as "head" tossing a coin <u>was</u> ½ before the coin was tossed)

BUT →

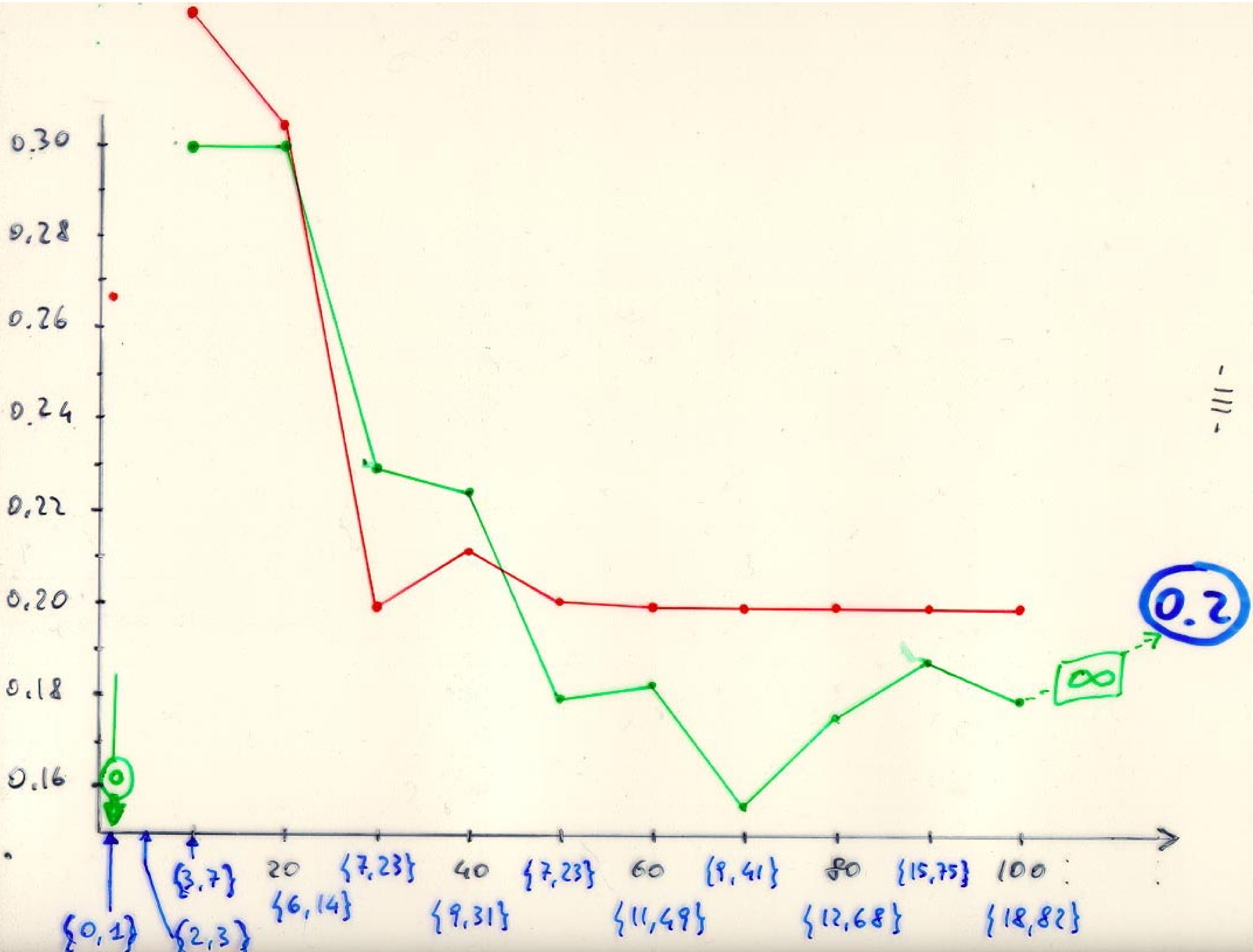
Now, the interesting question

is rather . "n+1"

$$P\left(E^{(101)} = E_1 \;\Big|\; \frac{\# E_1}{\# E_2} \simeq 0.2,\; n = 100\right)$$

and this is a different story!

| n | $f_n$ | Probability Bayesian | Probability frequentist |
|---|---|---|---|
| 0 | – | 50% | ? |
| 1 | 0/1 | 27% | 0 |
| 5 | 2/5 | 43% | 40% |
| 10 | 3/10 | 33% | 30% |
| 20 | 6/20 | 31% | 30% |
| 30 | 7/30 | 20.0% | 23.3% |
| 40 | 9/40 | 21.3% | 22.5% |
| 50 | 9/50 | 20.1% | 18.0% |
| 60 | 11/60 | 20.03% | 18.3% |
| 70 | 11/70 | 20.002% | 15.7% |
| 80 | 12/80 | 20.000% | 17.6% |
| 90 | 15/90 | 20.000% | 18.8% |
| 100 | 18/100 | 20.000% | 18.0% |
| | | ! | "n = ∞" |
| | | | 20% |


0.30
0.28
0.26
0.24
0.22
0.20
0.18
0.16
0
{0,1}
{2,3}
{3,7}
20
{6,14}
{7,23}
40
{9,31}
{7,23}
60
{11,49}
{9,41}
80
{12,68}
{15,75}
100
{18,82}
∞
0.2

Does it violate Bernoulli's theorem?
("low of large numbers")

$$\lim_{n \to \infty} f_n \to p$$

⇒ If we think to $n \to \underbrace{\infty}^{\text{situations}}$ in which we are exactly in the same status of knowledge given by $E^{(1)} = E_2$, i.e. we are interested to

$$P\left(E^{(2)} = E_1 \mid E^{(1)} = E_2\right)$$

there is nothing wrong:

$$\lim_{n \to \infty} f_n = 27\%$$

(try with a little Monte Carlo Simulation)

⇒ We should not confuse the event we are intersted to with

$$E^{(2)} = E_1 \mid H = H_1,$$

which is something different.