# Probabilistic interpretation of Peelle's pertinent puzzle and its resolution

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LA-UR-04-6723

# Overview

- Peelle's Pertinent Puzzle (1987) –
  paradoxical result produced by strong correlations in errors
- Probabilistic view of PPP
- Specific probabilistic model for PPP elucidates how correlations in errors arise
- Plausible experimental situation consistent with PPP result
- Other probabilistic interpretations of PPP statement
- Bayesian approach to coping with uncertainty in model
- PPP underlines the need for **how** uncertainties contribute to reporter data

# Peelle's pertinent puzzle

- Robert Peelle (ORNL) posed the PPP in 1987:
  Given two measurements of same quantity $x$:
  $m_1 = 1.5$; $m_2 = 1.0$ ,
  each with independent standard error of 10% ,
  and fully correlated standard error of 20% .
  Weighted average using least-squares is $x = 0.88 \pm 0.22$

- Peelle asks "under what conditions is this result reasonable?"
- By extension, if this not reasonable, what answer is appropriate?
- PPP is pertinent! – its effect has been present in nuclear data evaluation for decades
- Comment – PPP description of errors is ambiguous, which leads to numerous plausible interpretations

# Standard PPP solution

- The solution given in PPP is based on standard matrix equations for least-squares result:

  estimated value $\boldsymbol{x} = (\boldsymbol{G}^T \boldsymbol{C}^{-1} \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{C}^{-1} \boldsymbol{m}$

  covariance in estimate $\boldsymbol{V} = (\boldsymbol{G}^T \boldsymbol{C}^{-1} \boldsymbol{G})^{-1}$

  where the sensitivity matrix is $\boldsymbol{G} = [1.0 \;\; 1.0]$

  and the measurements are the vector $\boldsymbol{m} = [1.5 \;\; 1.0]^{\boldsymbol{T}}$

  with covariance matrix $\boldsymbol{C} = \begin{pmatrix} 1.5^2 * (0.1^2 + 0.2^2) & 1.5 * 1.0 * 0.2^2 \\ 1.5 * 1.0 * 0.2^2 & 1.0^2 * (0.1^2 + 0.2^2) \end{pmatrix}$

- Result is $x = 0.88 \pm 0.22$
- This result is smaller than both measurements, which seems implausible

# Probabilistic view of standard PPP solution

- Consider the probability density function (pdf) for the variables

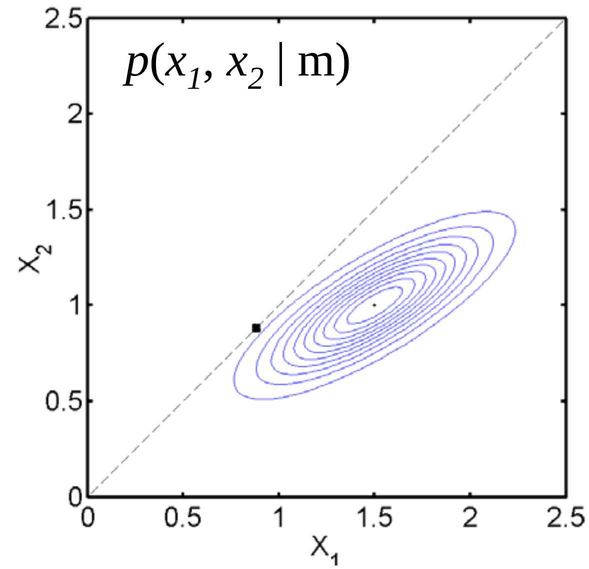
$$\boldsymbol{x} = [x_1 \quad x_2]^{\boldsymbol{T}}$$

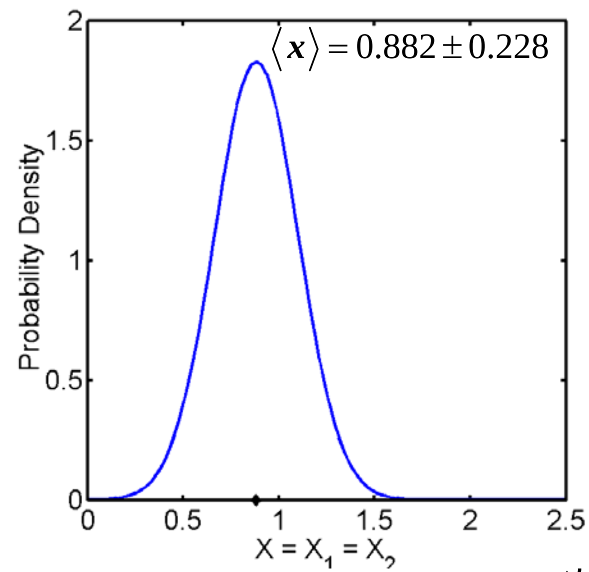
$$p(\boldsymbol{x} \mid \boldsymbol{m}) \propto \exp\left\{ -\frac{1}{2}^{\boldsymbol{T}} \left(\boldsymbol{x} - \boldsymbol{m}\right)^{\boldsymbol{T}} \boldsymbol{C}^{-1} \left(\boldsymbol{x} - \boldsymbol{m}\right) \right\}$$

where measurements are $\boldsymbol{m} = [1.5 \;\; 1.0]^{\boldsymbol{T}}$ and their covariance matrix is

$$\boldsymbol{C} = \begin{pmatrix} 1.5^2 * (0.1^2 + 0.2^2) & 1.5 * 1.0 * 0.2^2 \\ 1.5 * 1.0 * 0.2^2 & 1.0^2 * (0.1^2 + 0.2^2) \end{pmatrix}$$

- For $x = x_1 = x_2$ (diagonal of 2D pdf), $p(x|\boldsymbol{m})$ is normal distribution centered at 0.88



# Probabilistic model for additive error

- Represent common uncertainty in measurements by systematic additive offset Δ: $x = m_1 + \Delta; \quad x = m_2 + \Delta$
- Bayes law gives joint pdf for $x$ and Δ

$$p(x, \Delta \mid \boldsymbol{m}) = p(\boldsymbol{m} \mid x, \Delta)\, p(x)\, p(\Delta)$$

where priors $p(x)$ is uniform and $p(\Delta)$ assumed normal ($\sigma_\Delta = 0.2$)

- Writing $p(x, \Delta \mid \boldsymbol{m}) \propto \exp\{-\varphi\}$ and assuming normal distributions

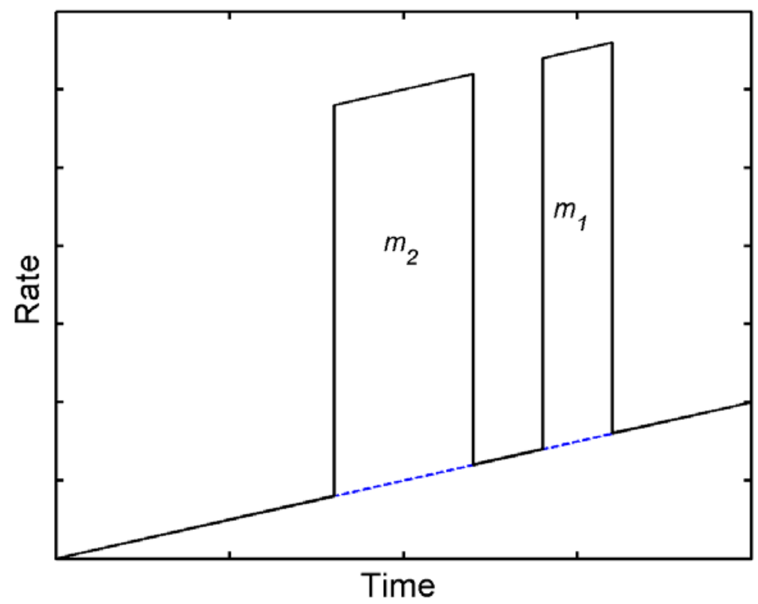
$$2\varphi = \frac{(x - m_1 - \Delta)^2}{\sigma_1^2} + \frac{(x - m_2 - \Delta)^2}{\sigma_2^2} + \frac{(\Delta - 1)^2}{\sigma_\Delta^2}$$

where $\sigma_1 = 0.1 * m_1; \quad \sigma_2 = 0.1 * m_2; \quad \sigma_\Delta = 0.2$

- Pdf for $x$ obtained by integration: $p(x \mid \boldsymbol{m}) = \int p(x, \Delta \mid \boldsymbol{m}) \mathrm{d}\Delta$
- This model exactly same as $p(\boldsymbol{x} \mid \boldsymbol{m}) \propto \exp\left\{ -\frac{1}{2}^{\boldsymbol{T}} (\boldsymbol{x} - \boldsymbol{m})^{\boldsymbol{T}} \boldsymbol{C}^{-1} (\boldsymbol{x} - \boldsymbol{m}) \right\}$

# Plausible experimental scenario

- Under what conditions is PPP result reasonable?
- Suppose that measurements made in intervals shown
- From experience with apparatus, we know that background increases linearly in time
- Background subtraction for $m_1$ is 1.5 times larger than for $m_2$ , which leads to stated covar. matrix
- For this scenario, previous model is appropriate, and PPP solution, 0.88, is correct answer



# Probabilistic model for normalization error

- Represent common uncertainty in measurements by systematic error in normalization factor *c*: $x = m_1 / c; \quad x = m_2 / c$
- Following same development as before, where prior $p(c)$ assumed normal with expected value of 1 and $\sigma_c = 0.2$
- Writing $p(x, c \mid \boldsymbol{m}) \propto \exp\{-\varphi\}$

$$2\varphi = \frac{(cx - m_1)^2}{\sigma_1^2} + \frac{(cx - m_2)^2}{\sigma_2^2} + \frac{(c-1)^2}{\sigma_c^2}$$

where $\sigma_1 = 0.1 * m_1; \quad \sigma_2 = 0.1 * m_2; \quad \sigma_c = 0.2$

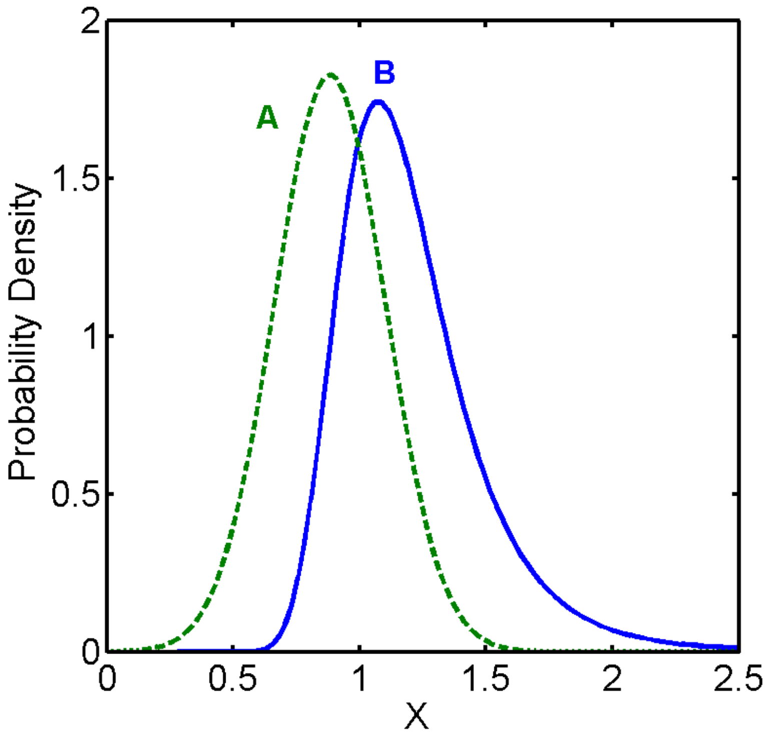
- Divide $p(cx, c)$ by Jacobian $J = 1/c$ to get $p(x, c)$
- $p(x)$ obtained by numerical integration: $p(x \mid \boldsymbol{m}) = \int p(x, c \mid \boldsymbol{m}) \mathrm{d}c$
- This approach promoted by D. Smith (1991)

# Probabilistic model for normalization error

- Compare pdfs for two models for correlated effect:
  A – additive offset
  B – normalization factor

- Observe significant difference in two results
- Emphasizes need to know which kind of effect leads to correlated error

- Probabilistic model is capable of handling various known effects



# Other models to include normalization error

- In previous model, because the normalization factor *c* is a scale parameter, one may argue that prior on *c* should be a log-normal distribution, i.e., a normal distribution in log(*c*)
- Then, writing $p(x,c\,|\,\boldsymbol{m}) \propto \exp\{-\varphi\}$

$$2\varphi = \frac{(cx - m_1)^2}{\sigma_1^2} + \frac{(cx - m_2)^2}{\sigma_2^2} + \frac{\log^2(c)}{\sigma_c^2}$$

- Jacobian $J = 1$, so $p(cx, c)$ is same as $p(x, c)$
- $p(x)$ obtained by numerical integration: $p(x\,|\,\boldsymbol{m}) = \int p(x,c\,|\,\boldsymbol{m})\,\mathrm{d}c$
- Resulting pdf is slightly different than for previous model

- Another approach is to take logarithm of data, transforming multiplicative normalization error to additive error
  – formulas for linear, additive errors may be applied

# Probabilistic view of Chiba-Smith solution

- Assume the correlated error is to be applied to inferred *x* value:

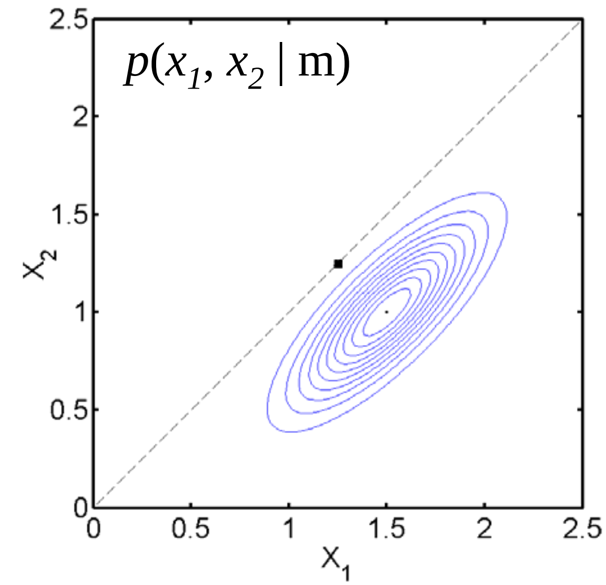
$$\boldsymbol{C} = \begin{pmatrix} \hat{\boldsymbol{x}}^2(0.1^2+0.2^2) & \hat{\boldsymbol{x}}^2 0.2^2 \\ \hat{\boldsymbol{x}}^2 0.2^2 & \hat{\boldsymbol{x}}^2(0.1^2+0.2^2) \end{pmatrix};$$

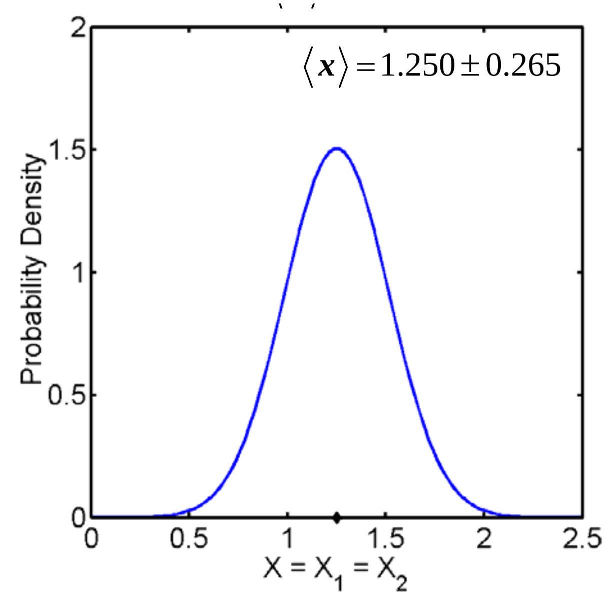
$$\hat{\boldsymbol{x}} = \frac{\left(m_1/\rho_1^2 + m_2/\rho_2^2\right)}{\left(1/\rho_1^2 + 1/\rho_2^2\right)}$$

- Plot shows $p(x_1, x_2 \mid \boldsymbol{m})$

$$p(\boldsymbol{x}\mid\boldsymbol{m}) \propto \exp\left\{-\frac{1}{2}^{\boldsymbol{T}}\left(\boldsymbol{x}-\boldsymbol{m}\right)^{\boldsymbol{T}}\boldsymbol{C}^{-1}\left(\boldsymbol{x}-\boldsymbol{m}\right)\right\}$$

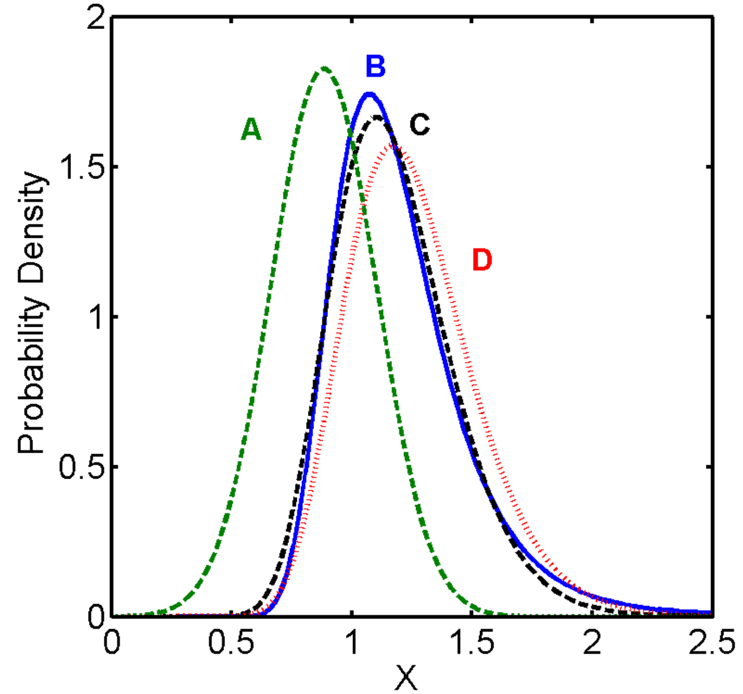
- For $x = x_1 = x_2$ (diagonal of 2D pdf), $p(x|\boldsymbol{m})$ is normal distribution centered at 1.25



# Results for various probabilistic models

- Plot shows pdfs for *x* for various models
  - additive error - PPP (dashed, A)
  - normalization error (solid, B)
  - normalization error with log-normal prior (dashed, C)
  - using logarithm of data (dotted, D)




- Table summarizes results
- PPP solution substantially different from others
- Those based on multiplicative normalization error are similar

| Method | *xmax* | *xmean* | *σx* |
|---|---|---|---|
| A – PPP - additive | 0.882 | 0.882 | 0.228 |
| B - normalization | 1.074 | 1.200 | 0.276 |
| C – " with log prior | 1.101 | 1.177 | 0.253 |
| D - log transform | 1.171 | 1.252 | 0.267 |

# Which model should we use?

- Ambiguity in specifying source of correlation leads to uncertainty about which model to use
- Bayesian approach can handle model uncertainty

$$p(x \mid \boldsymbol{m}) = \int p(x, M \mid \boldsymbol{m})\, \mathrm{d}M$$

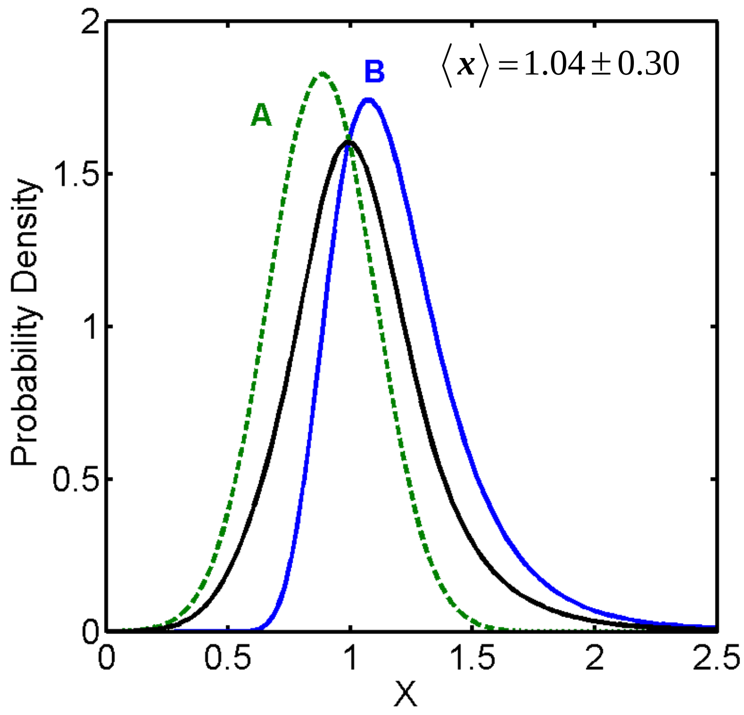
$$= \int p(x \mid \boldsymbol{m}, M)\, p(M)\, \mathrm{d}M$$

$$= \frac{1}{2} p(x \mid \boldsymbol{m}, M_1) + \frac{1}{2} p(x \mid \boldsymbol{m}, M_2)$$

  - for two equally likely models $M_1$ and $M_2$

- Answer is average of both pdfs!!

$$x = 1.04 \pm 0.30$$




solid black line is average of A and B

# Conclusions

- PPP result is consistent with plausible experimental scenario
  - in which correlated (systematic) error contributes additively to result
- Ambiguous statement of the PPP leads to other interpretations
  - some of which yield more plausible answers
- Analysts need better information to analyze data without guessing

- Probabilistic modeling can cope with various known uncertainty effects

# Conclusions

- **Experimenters – please provide measurement details**
- Some of the details needed:
  - specify standard errors as precisely as possible, indicating where uncertainties in their assessment lie
  - specify components in uncertainties and whether they are
    - independent, or correlated, e.g., systematic errors
    - given relative to measured quantities or inferred values
    - additive (background subtraction) or multiplicative (normalization)
- **Correlation matrix by itself may not be enough**

- Another issue in PPP is inconsistency between two measurements: one can cope with this discrepancy by introducing notion that the true errors may differ from quoted errors, i.e., treatment of outliers