Inferences from Rossi traces

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Overview of presentation

- Goal - assess uncertainty in alpha curves
- Rossi technique for recording time-dependent signals
- Uncertainties in reading Rossi traces $\Rightarrow$ likelihood
- Model - alpha as function of time, then calculate data
- Bayesian data analysis -
  posterior provides inference about model parameters
- Markov Chain Monte Carlo technique -
  display and quantify uncertainty distribution
- Uncertainties in alpha curve

Acknowledgements

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  - Kent Croasdell, Layle Zongker, Rick Collingsworth, Tom Tunnel
- General discussions
  - Tom Gorman, Jamie Langenbrunner, Shane Reese, Greg Cunningham

Alpha - measure of criticality

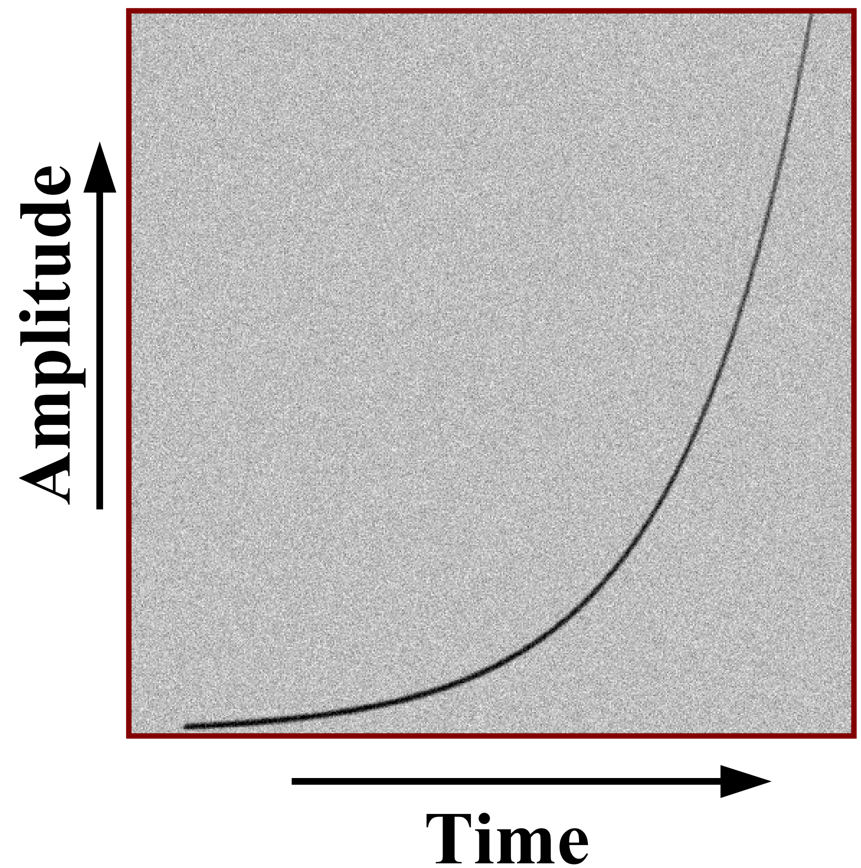
- Assembly of radiographic, fissionable material can become critical, i.e. neutron fluxes can grow exponentially
- If $y(t)$ is neutron flux as function of time, the "Rossi" α is measure of extent of criticality:

$$\alpha(t) = \frac{1}{y}\frac{dy}{dt} = \frac{d(\ln y)}{dt}$$

- Objective is to infer $\alpha(t)$ from measurements of "Rossi" traces, $y(\cos(t))$
- Use of Bayesian analysis, coupled with MCMC technique, allows full assessment of uncertainties in inference, including effect of systematic uncertainties

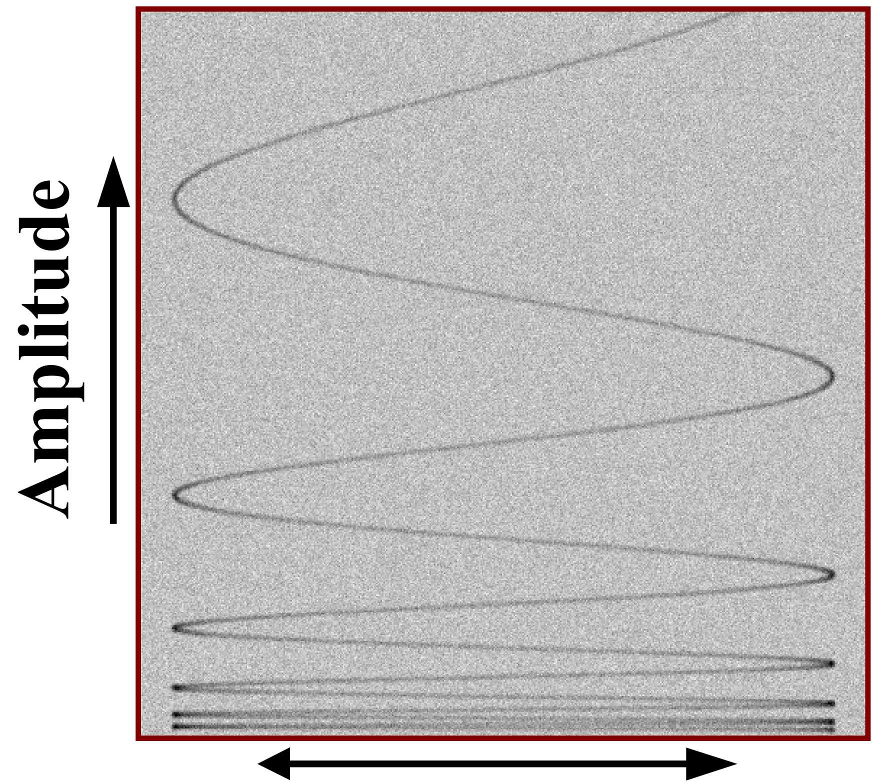
Standard amplitude vs. time recording

- Objective is to accurately record an exponentially rising signal
- Standard technique is to photograph CRT screen
  - horizontal sweep linear in time
  - signal amplitude vertical
- CRT nonlinearities and errors in orientation lead to errors in measurement of amplitude vs. time



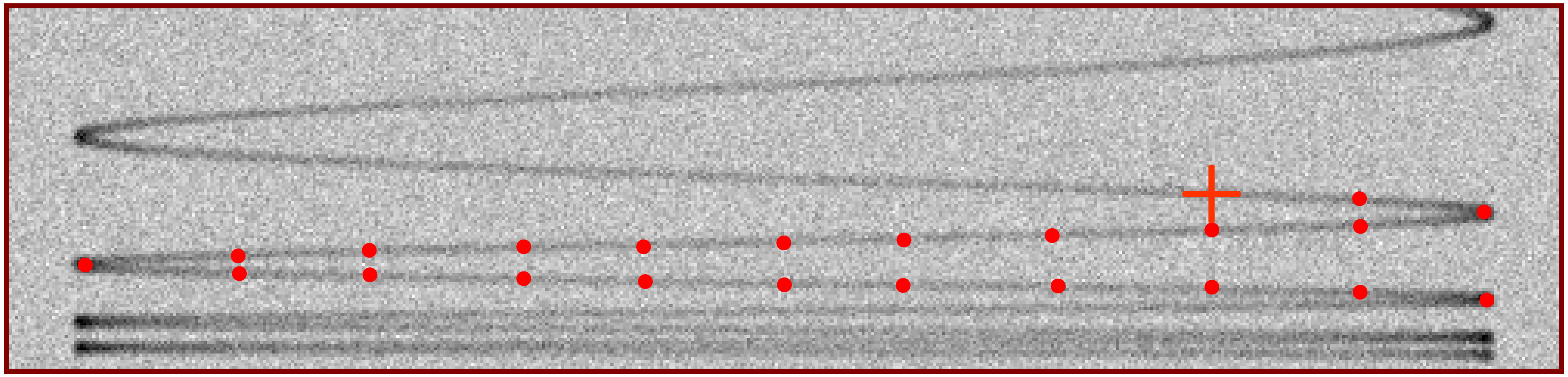
The Rossi technique

- Rossi technique - photograph oscilloscope screen
  - horizontal sweep is driven sinusoidally in time
  - signal amplitude vertical
- Records rapidly increasing signal while keeping trace in middle of CRT, which minimizes nonlinearities



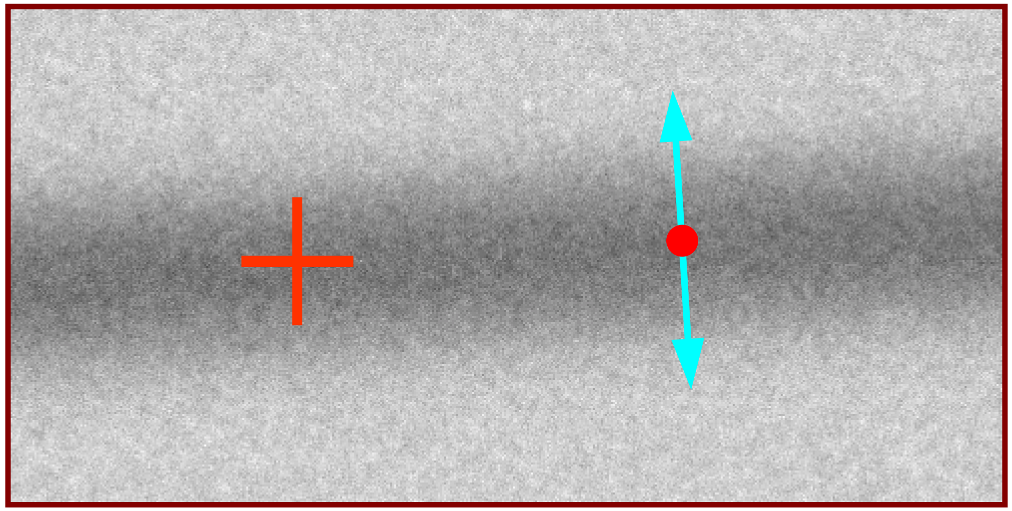
$$x = x_R \cos(2\pi f_R t + \phi_0)$$

Reading a Rossi trace



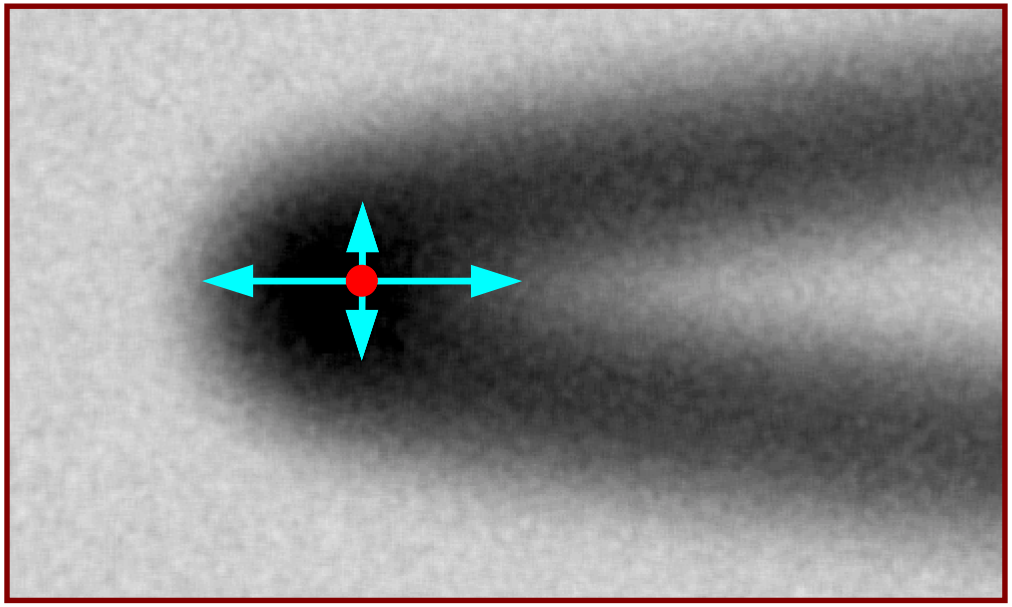
- Technician reads points by centering cross hairs of a reticule on trace; computer records positions, $\{x_i, y_i\}$
- Points are read:
  - approximately evenly spaced along trace
  - otherwise arbitrary placement along curve, except at peaks

Uncertainties in Rossi readings



- Readings obtained by centering cross hairs on trace
- Uncertainties in location of trace
  - depend on width of trace and signal-to-noise ratio
  - perpendicular to trace (position along trace arbitrarily chosen)

Uncertainties in Rossi readings



- Near peaks, intensity varies because it depends on speed at which electron beam moves over CRT phosphor
- Uncertainties in placement
  - mainly in horizontal direction
  - larger than in regions where trace is more straight

Cubic spline expansion of alpha curve

– Expand $\alpha(t)$ in terms of basis functions:

$$\alpha(t) = \sum_k a_k \, \phi\left[\frac{t - t_k}{\Delta t}\right]$$

where

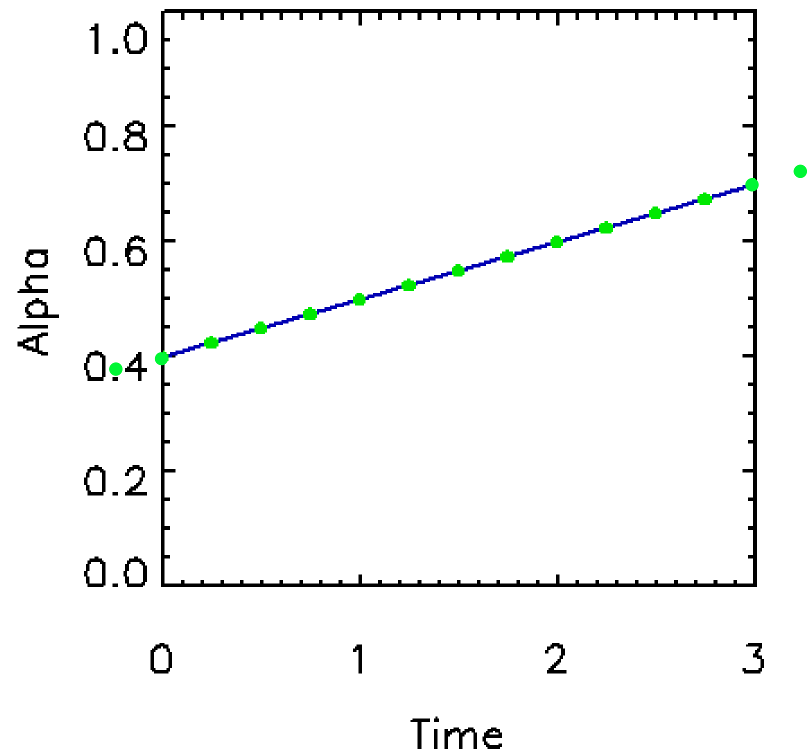
- a_k is the expansion coefficient,
- ϕ is a spline basis function,
- t_k is the position of the kth knot
- Δt is the knot spacing

– Use 15 evenly-space knots

- spacing chosen on basis of limited bandwidth of signal y
- two are outside data interval to avoid special end conditions

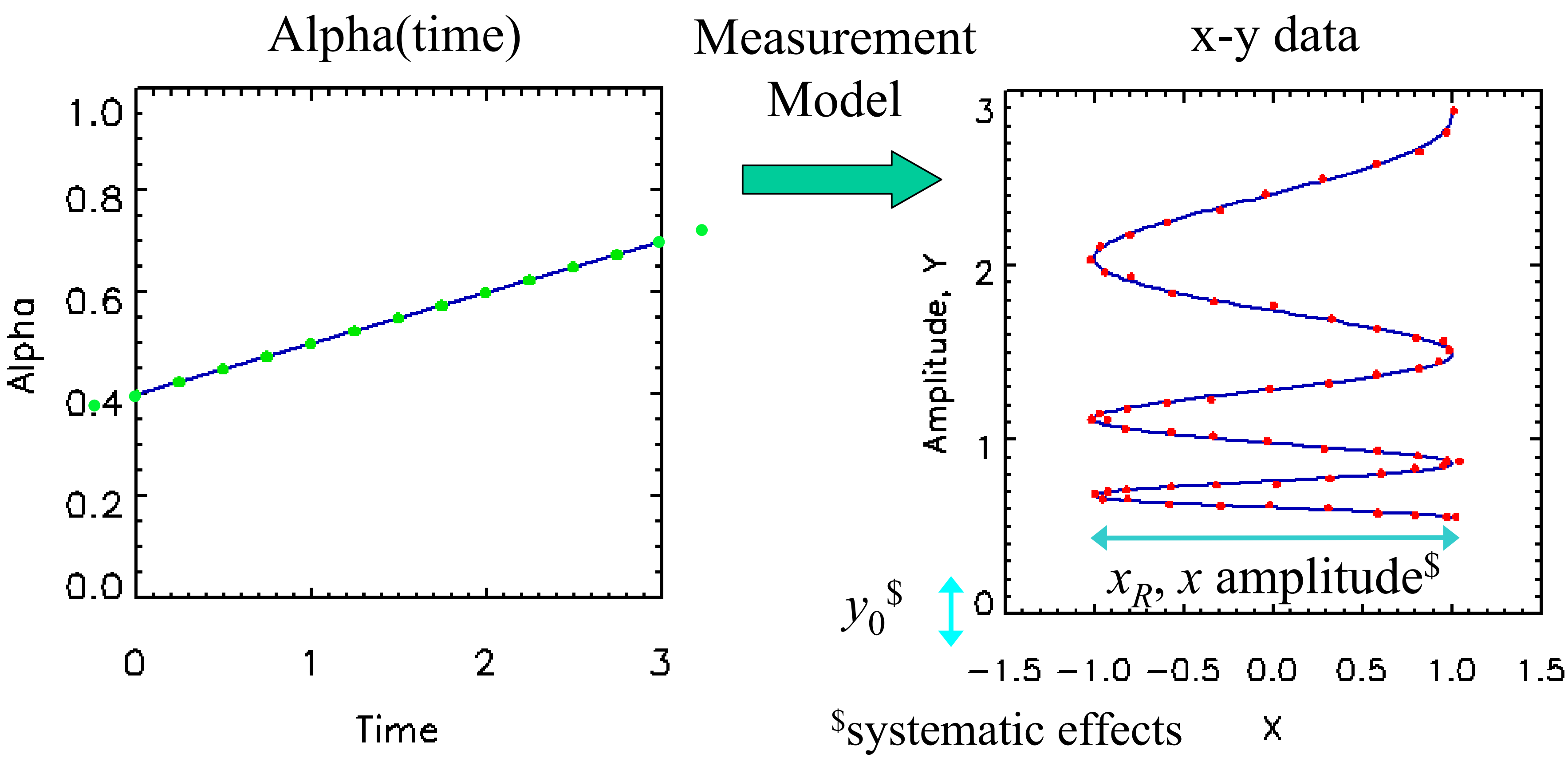
– Parameters a_k are to be determined

Alpha(time)

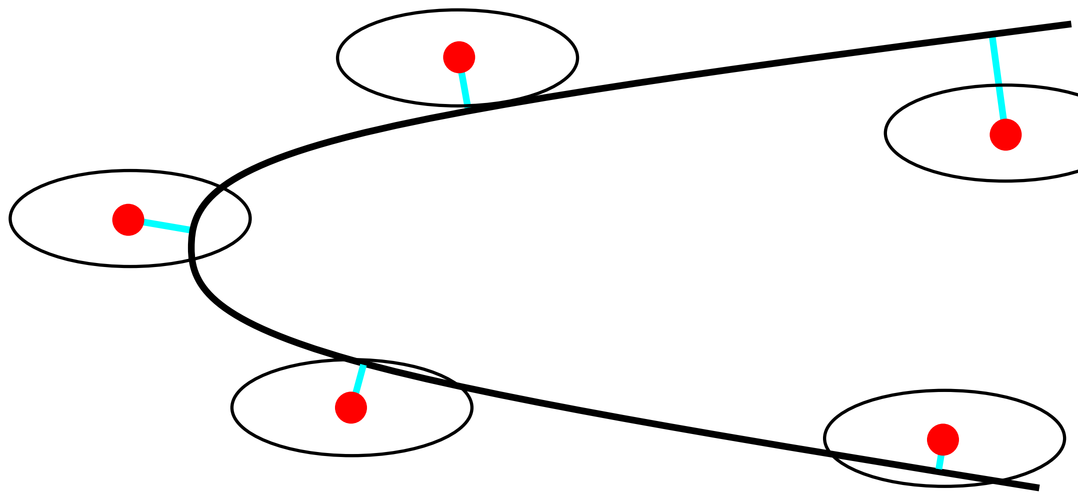


Modeling the Rossi data

- $\alpha(t)$ represented as cubic spline
- measurement model predicts data
- include systematic effects of measurement system, y_0 and x_R



Likelihood model - uncertainties in Rossi data



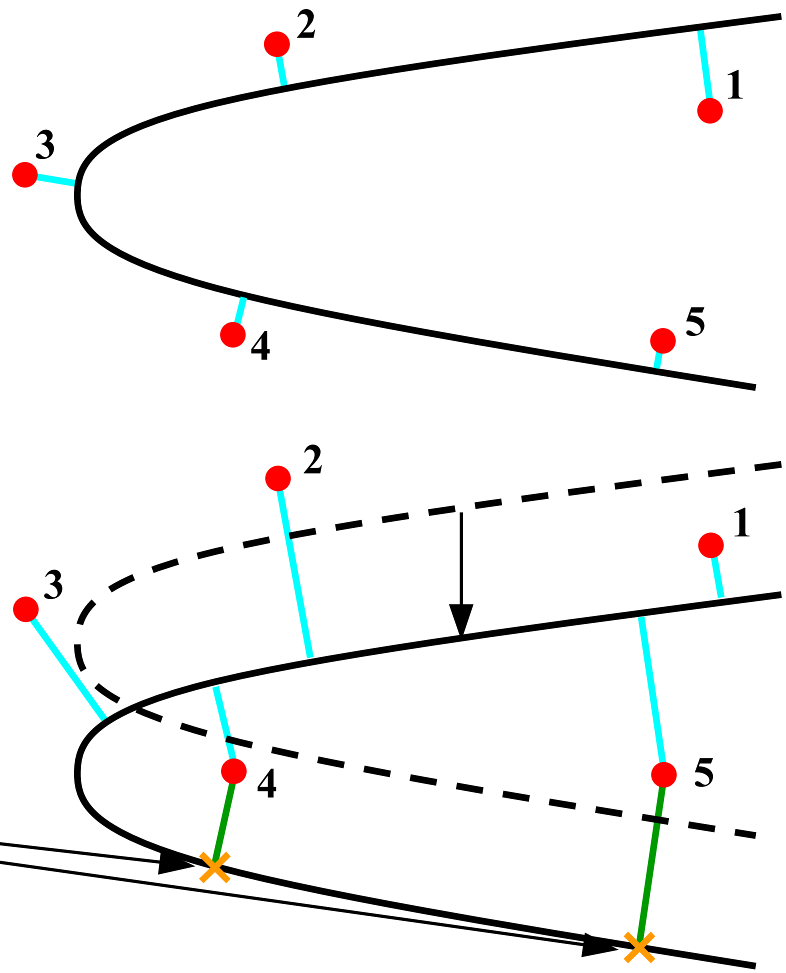
– minus-log-likelihood, p(**d**|**a**), for measured point (x_{exp}, y_{exp}):

$$\Delta \frac{\chi^2}{2} = \frac{(x_{exp} - x'_{model})^2}{2\sigma_x^2} + \frac{(y_{exp} - y'_{model})^2}{2\sigma_y^2}$$

where (x'_{model}, y'_{model}) is the model point closest to (x_{exp}, y_{exp})

Likelihood model

- Potential problem with taking distance to nearest point on curve
  - points may be measured in a specific order
  - when curve is shifted, order of points may get mixed up; leads to discontinuities in Λ
- Potential remedy is to constrain reference points on curve to maintain order of measured points



Cubic spline expansion

- Expand $\alpha(t)$ in terms of basis functions:

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where a_k is the expansion coefficient,
ϕ is a spline basis function,
t_k is the position of the kth knot, and
Δt is the knot spacing

- Parameters a_k are to be determined

Bayesian analysis of an experiment

- The pdf describing uncertainties in model parameter vector **a**, called **posterior**:
  - $p(\mathbf{a}|\mathbf{d}) \sim p(\mathbf{d}|\mathbf{d}^*)\, p(\mathbf{a})$ (Bayes law)
    where **d** is vector of measurements, and
    d*(a) is measurement vector predicted by model
  - $p(\mathbf{d}|\mathbf{d}^*)$ is likelihood, probability of measurements **d** given the values **d*** predicted by simulation of experiment
  - $p(\mathbf{a})$ is prior; summarizes previous knowledge of **a**
  - "best" parameters estimated by
    - maximizing posterior (called MAP solution)
    - mean of posterior
  - uncertainties in **a** are fully characterized by $p(\mathbf{a}|\mathbf{d})$

Smoothness constraint

- Cubic splines tend to oscillate in some applications
- Smoothness of $\alpha(t)$ can be controlled by minimizing

$$S(\alpha) = T^3 \int \left| \frac{d^2\alpha}{dt^2} \right|^2 dt$$

where T is the interval; T^3 factor removes T dependence

- Smoothness can be incorporated in Bayesian context by setting prior on spline coefficients to

$$-\log p(\mathbf{a}) = \lambda\, S(\alpha(\mathbf{a}))$$

- Hyperparameter λ can be determined in Bayesian approach by maximizing $p(\lambda|\mathbf{d})$

Bayesian posterior

- The full Bayesian posterior for spline coefficients in Rossi analysis is

$$- \log(p(\mathbf{a}|\mathbf{d})) = \frac{(y_0 - y_{0,\text{model}})^2}{2\sigma_y^2} + \sum \frac{(x_{\text{exp}} - x'_{\text{model}})^2}{2\sigma_x^2} + \frac{(y_{\text{exp}} - y'_{\text{model}})^2}{2\sigma_y^2} + \lambda S$$

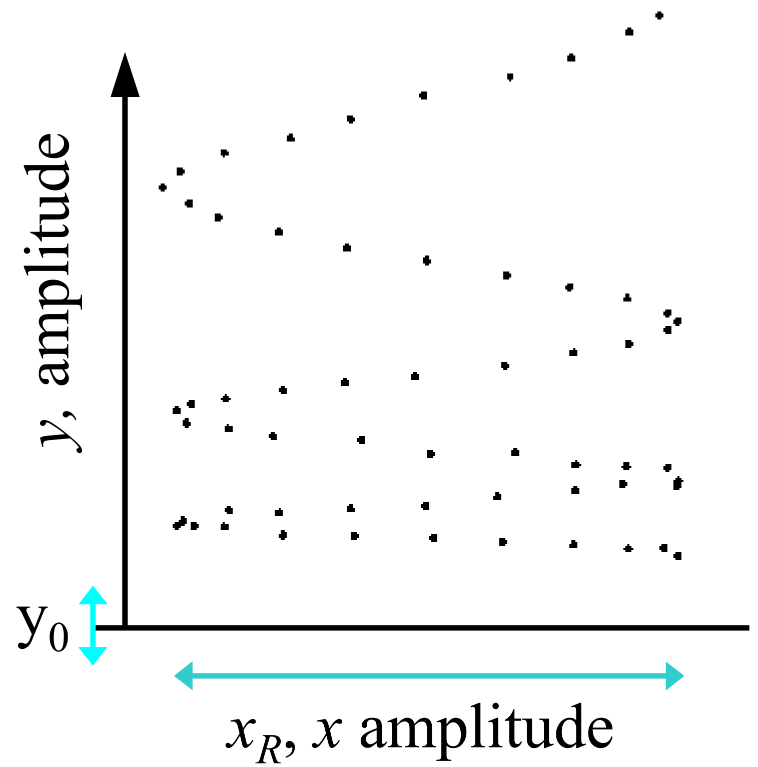
where first term comes from baseline y_0 measurement, second from (x, y) measurements of Rossi trace, last from prior on smoothness

- All inferences about $\alpha(t)$ are based on this posterior
- Data **d**: $\{x_i , y_i\}$, points from Rossi trace; y_0, y baseline
- Parameters **a**: $\{a_k\}$, knot coefficients; y_0, y baseline; x_R, amplitude of Rossi sweep

Likelihood model

- Log-likelihood contributions:
  - sum over all measured $(x_{\mathrm{exp}}, y_{\mathrm{exp}})$ points
  - measured baseline, y_0
- Model parameters:
  - Rossi frequency and t_0
  - y_0, baseline for y amplitude $^{\$}$
  - x_R, x amplitude $^{\$}$
  - $y(t)$ modeled as cubic spline with equal node spacing, chosen on basis of bandwidth of signal

    $^{\$}$systematic effects



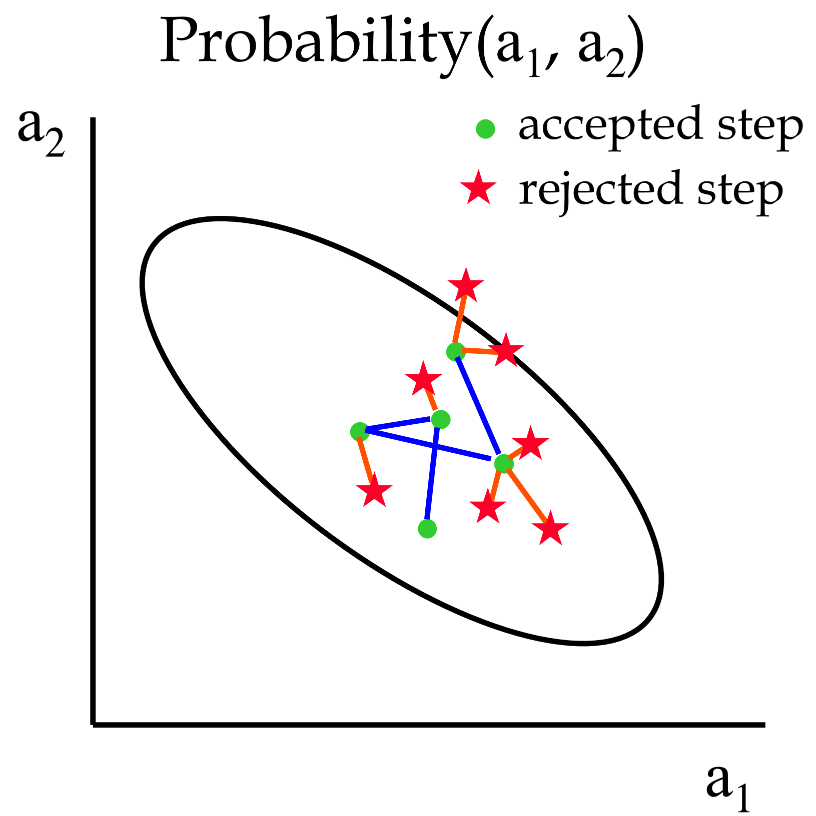
Parameter uncertainties via MCMC

- Posterior $p(\mathbf{a}|\mathbf{d})$ provides full uncertainty distribution for **a**
- Markov Chain Monte Carlo (MCMC) algorithm generates a random sequence of parameters that sample $p(\mathbf{a}|\mathbf{d})$
  - results in plausible set of parameters $\{\mathbf{a}\}$
  - variation in plausible set of **a** is representative of uncertainties
  - second moments of parameters can be used to estimate covariance matrix **C**
- MCMC advantages
  - can be applied to any pdf, not just Gaussians
  - automatic marginalization over nuisance variables
- MCMC disadvantage
  - potentially calculationally demanding

Markov Chain Monte Carlo

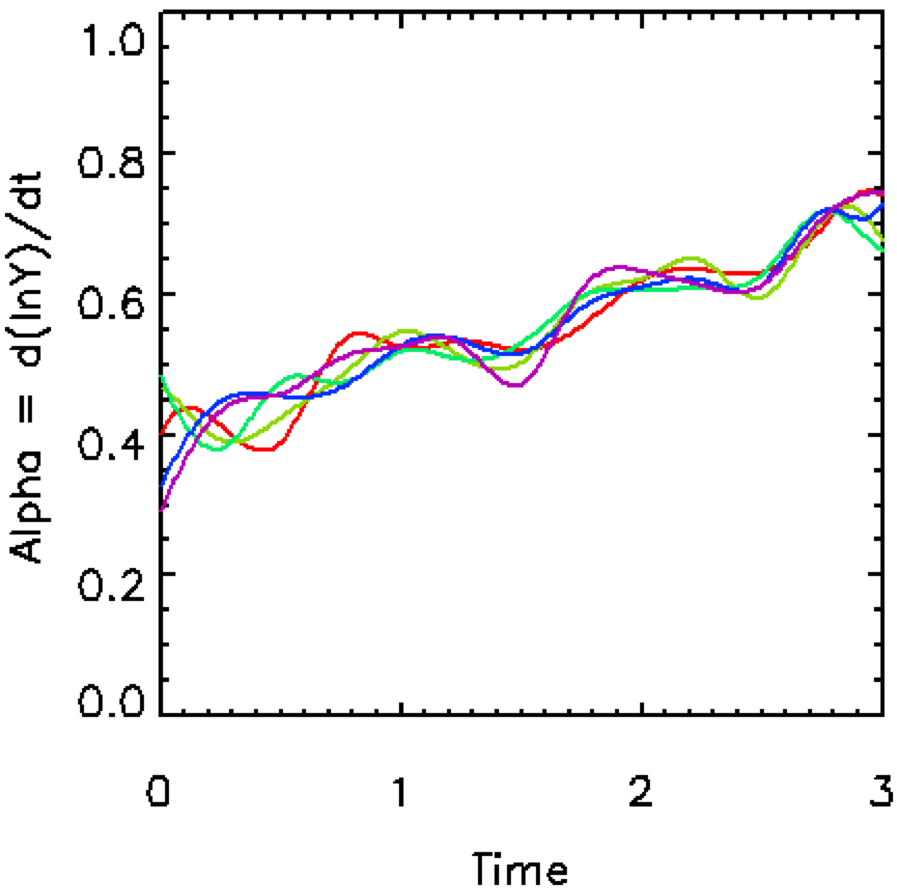
Generates sequence of random samples from an arbitrary probability density function

- Metropolis algorithm:
  - draw trial step from symmetric pdf, i.e., $T(\Delta \mathbf{a}) = T(-\Delta \mathbf{a})$
  - accept or reject trial step
  - simple and generally applicable
  - relies only on calculation of target pdf for any **a**



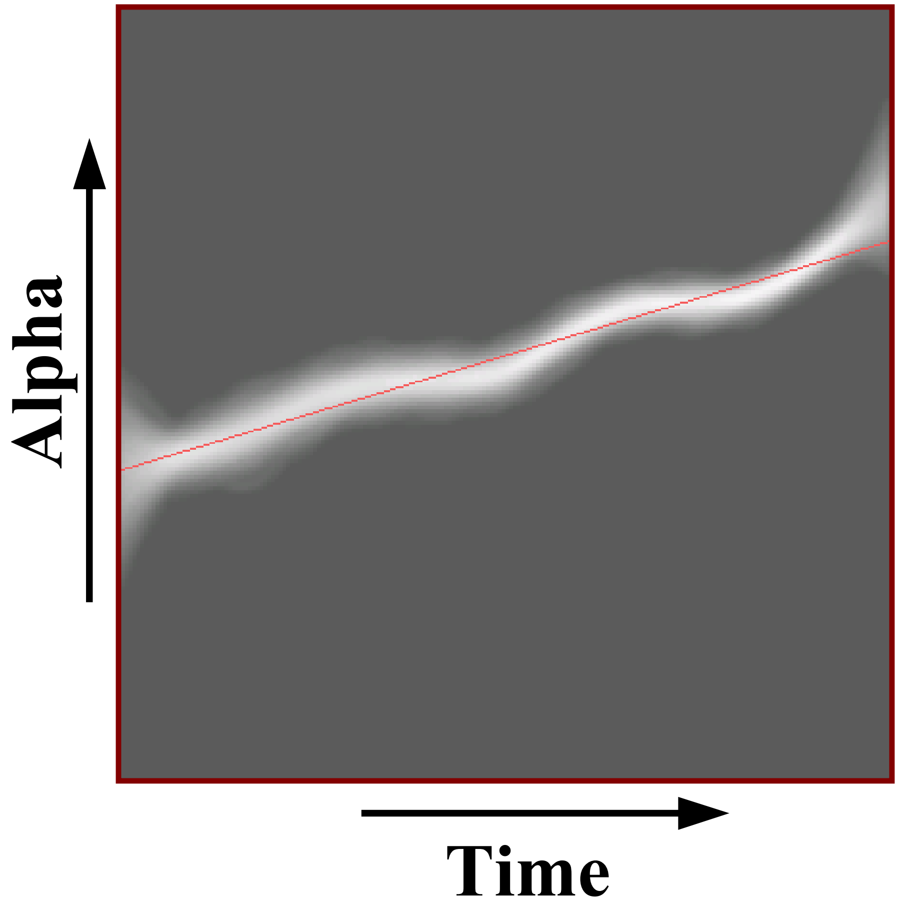
MCMC - alpha uncertainty

- MCMC samples from posterior
  - plot shows several $\alpha(t)$ curves consistent with data
  - uncertainties in model visualized as variations in curves
- Smoothness parameter, $\lambda = 0.04$



MCMC - Alpha

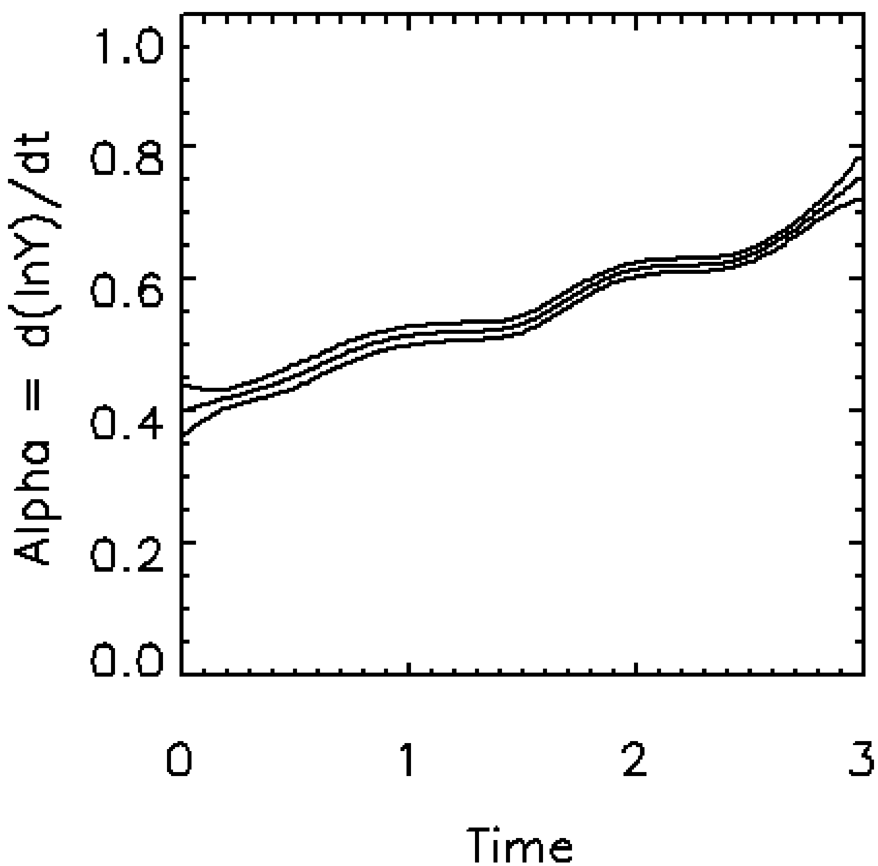
- For MCMC sequence with 10^5 samples, image shows accumulated MCMC curves in alpha domain
- Effectively shows PDF for uncertainty distribution in alpha, estimated from data
- However, does **not** show correlations between uncertainties at two different times, as do individual MCMC samples



$\lambda = 0.4$ (best value)

MCMC - Alpha

- Interpreting accumulated alpha curve as a PDF, one can estimate $\alpha(t)$ in terms of
  - posterior mean
  - posterior max. (MAP estimate)
- Or characterize uncertainties
  - standard deviations
  - covariance matrix (correlations)
  - credible intervals (envelope)
- Plot on right shows
  - posterior mean
  - posterior mean +/- standard dev. (one standard dev. envelope)

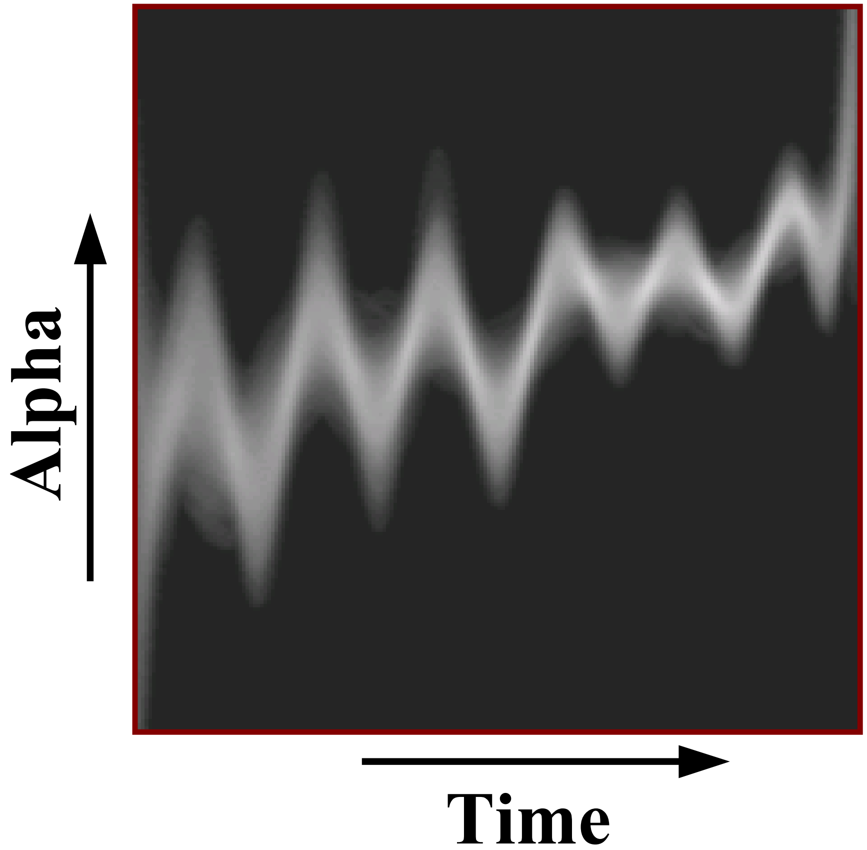


$\lambda = 0.4$ (best value)

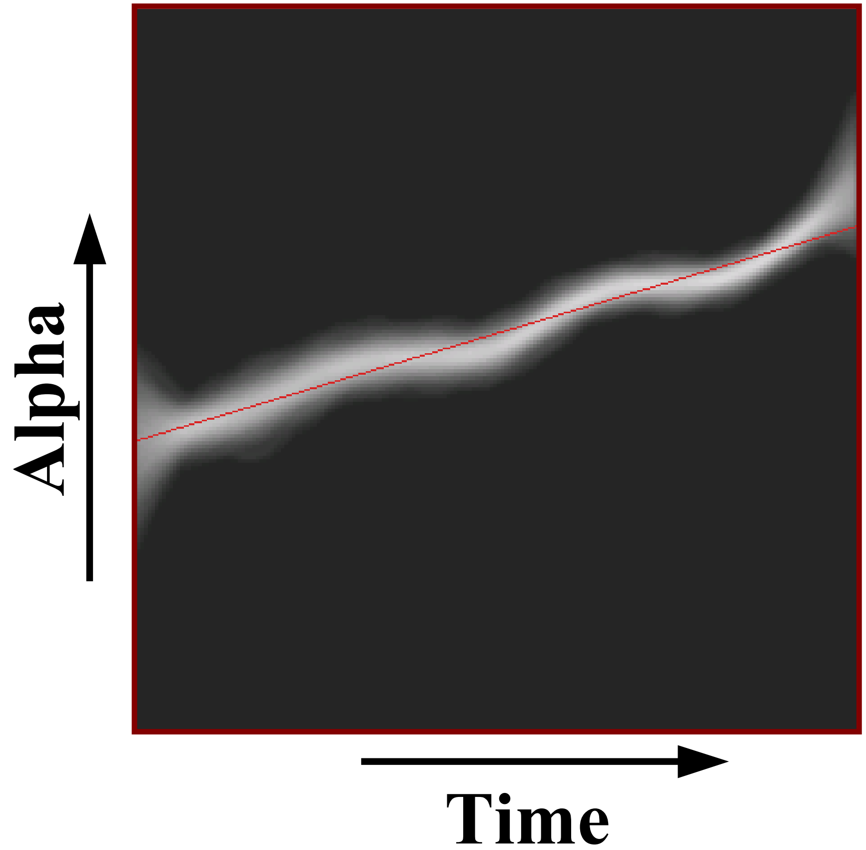
Effect of smoothing prior

Splines' tendency to oscillate is controlled by smoothing prior

$\lambda = 0.004$ (minimal prior)

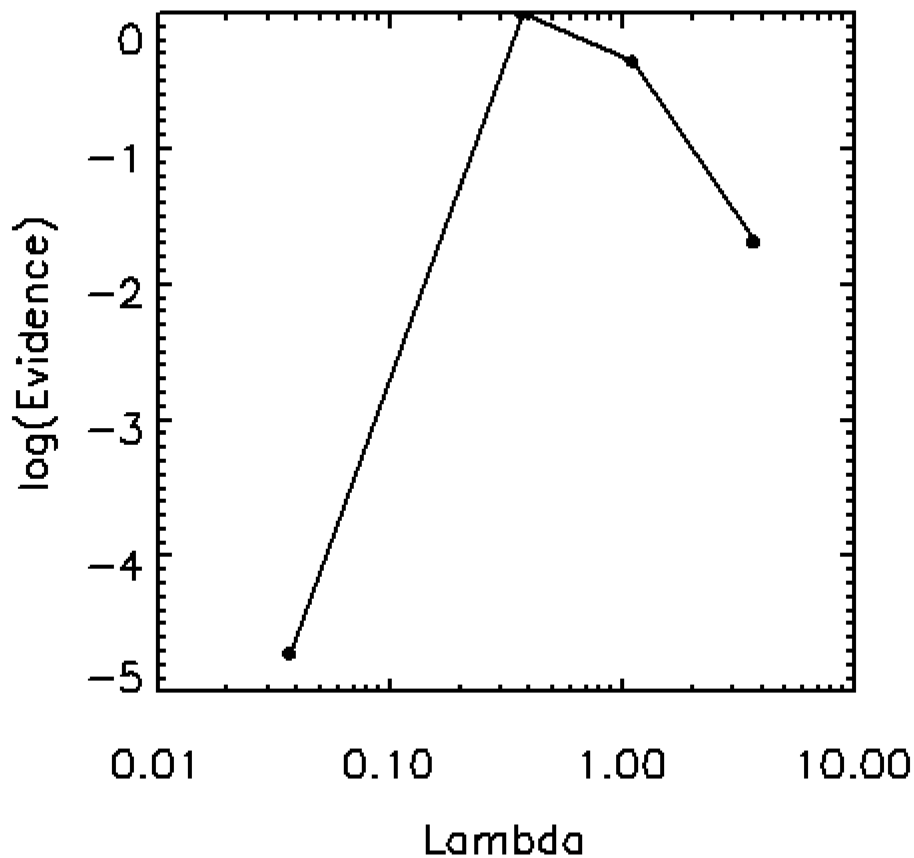


$\lambda = 0.4$ (best value)



Choice of the strength of smoothing prior

– Hyperparameter λ controls strength of smoothing prior
– chosen by maximizing $p(\lambda|\mathbf{d})$, which is proportional to the evidence
$p(\mathbf{d}|\lambda) = \int p(\mathbf{a})\, p(\mathbf{d}|\mathbf{a}, \lambda)\, d\mathbf{a}$
(probability of data, given model and λ)
– integral often approximated as peak value of integrand times its volume, given by the determinant of the covariance matrix



best value: λ = 0.4

MCMC - Autocorrelation and Efficiency

- In MCMC sequence, subsequent parameter values are usually correlated
- Degree of correlation quantified by autocorrelation function:

$$\rho(l) = \frac{1}{N} \sum_{i=1}^{N} y(i) y(i-l)$$

where *y(x)* is the sequence and *l* is lag

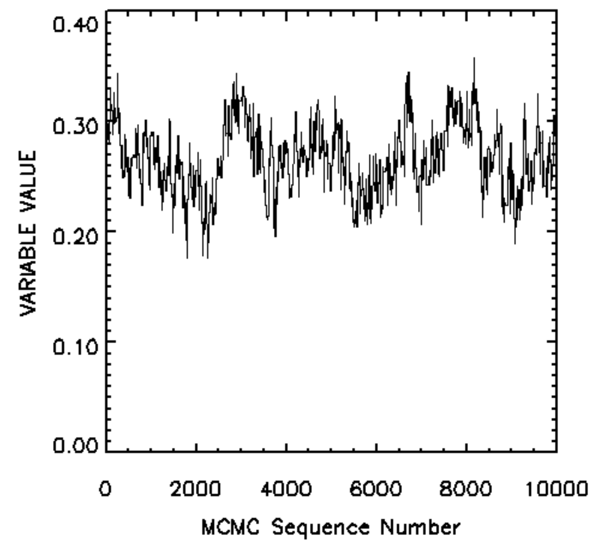
- For Markov chain, expect exponential

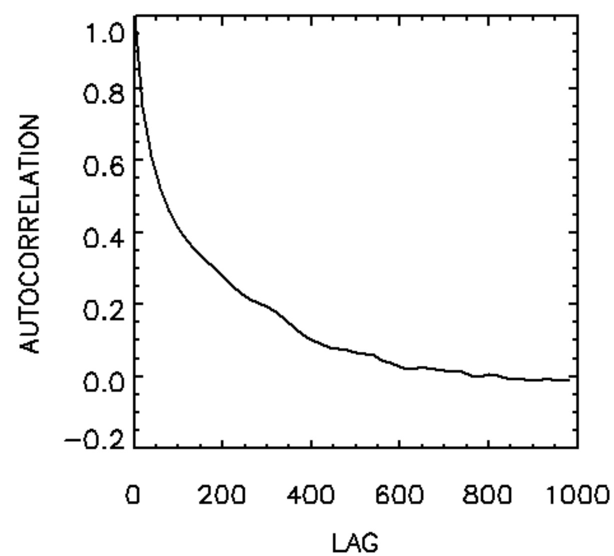
$$\rho(l) = \exp\left[-\left|\frac{l}{\lambda}\right|\right]$$

- Sampling efficiency is

$$\varepsilon = [1 + 2\sum_{l=1}^{\infty} \rho(l)]^{-1} = \frac{1}{1+2\lambda}$$

- For sequence shown, $\lambda = 170$, $\varepsilon = 0.003$





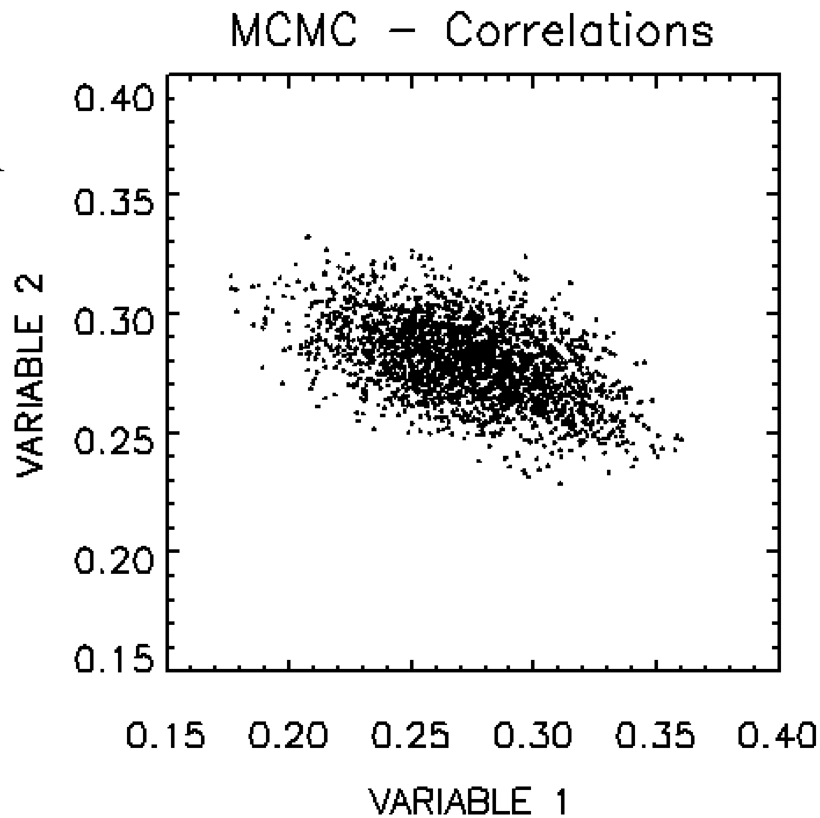
MCMC - Correlations among Uncertainties

– MCMC sequence can quantify correlations between uncertainties in various parameters

– Covariance between parameters:

$$[C]_{jk} = \frac{1}{N}\sum_{i=1}^{N} a_j(i)\, a_k(i)$$

where $a_j(i)$ is the value of the jth parameter at the ith sequence step



MCMC - Issues

- Choice of PDF for trial steps in parameters
  - desire improved efficiency in calculation
  - would like to incorporate correlations in posterior
- Burn in
  - may need to run MCMC for awhile to get in operating region of posterior distribution
- Convergence of sequence to true PDF
  - validity of estimated properties of parameters (covariance)
  - accuracy of same

Conclusions

- Bayesian analysis is a useful way to analyze experimental data in terms of models
- MCMC provides good tool for exploring the posterior and hence in drawing inferences about model

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