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Bayesian approach to limited-angle reconstruction in computed tomography

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An arbitrary source function cannot be determined fully from projection data that are limited in number and range of viewing angle. There exists a null subspace in the Hilbert space of possible source functions about which the available projection measurements provide no information. The null-space components of deterministic solutions are usually zero, giving rise to unavoidable artifacts. It is demonstrated that these artifacts may be reduced by a Bayesian maximum *a posteriori* (MAP) reconstruction method that permits the use of significant *a priori* information. Since normal distributions are assumed for the *a priori* and measurement-error probability densities, the MAP reconstruction method presented here is equivalent to the minimum-variance linear estimator with nonstationary mean and covariance ensemble characterizations. A more comprehensive Bayesian approach is suggested in which the ensemble mean and covariance specifications are adjusted on the basis of the measurements.

INTRODUCTION

The problem of obtaining an artifact-free computed-tomographic (CT) reconstruction from projection data that are limited in number and possibly in angular coverage is, in general, a difficult one to solve. This difficulty arises from a fundamental limitation inherent in incomplete data sets. It is seen that this limitation may be viewed as arising from an essential lack of information about the unknown source function, that is, its null-space components.[1] The Bayesian approach allows one to incorporate *a priori* information about the source function based on the properties of the ensemble of source functions realizable in the specified imaging situation. If the *a priori* information is specific enough, reasonable estimates of the null-space components of the source function can be obtained, thereby reducing the artifacts in the reconstruction. The results of this maximum *a posteriori* (MAP) method[2] are compared with the fit and iterative reconstruction (FAIR) technique.[3] We propose in the discussion that the incorporation of the Bayesian approach in the FAIR technique can provide a more-flexible algorithm.

MEASUREMENT SPACE–NULL SPACE

The CT problem may be stated as follows: Given a finite set of projections of a function of two dimensions $f(x, y)$ with compact support, obtain the best estimate of that function. The projections may generally be written as a weighted two-dimensional (2-D) integral of $f(x, y)$,

$$g_i = \iint h_i(x, y) f(x, y) \mathrm{d}x \mathrm{d}y, \qquad (1)$$

where the h_i are the weighting functions and $i = 1, 2, \ldots N$ for N individual measurements. We refer to the h_i as response functions. In the CT problem the h_i typically have large values within a narrow strip and small or zero values outside the strip. If the h_i are unity within a strip and zero outside, Eq. (1) becomes a strip integral. For zero strip width, it becomes a line integral. These last two cases are recognized as idealizations of the usual physical situation. The generality of Eq. (1) allows it to represent closely actual physical measurements since it can take into account response functions that vary with position. Note that Eq. (1) is applicable to any discretely sampled, linear-imaging system. Thus the concept of null space and the Bayesian methods proposed for overcoming its limitations are relevant to a large variety of image-restoration problems.

The unknown function $f(x, y)$ is usually restricted to a certain class, for example, the class of all integrable functions with compact support. Consider the Hilbert space of all acceptable functions and assume that all the h_i belong to that space. Equation (1) is an inner product of h_i with f. Thus the measurements g_i may be thought of as a projection of the unknown vector f onto the response vector h_i. Only those components of f that lie in the subspace spanned by the set of all h_i contribute to the measurements. We call this subspace the measurement space. The components of f in the remaining orthogonal subspace, the null space, do not contribute to the measurements. Hence the null-space contribution to f cannot be determined from the measurements alone. Since the deterministic (measurement) subspace of f is spanned by the response functions, it is natural to expand the estimate of f in terms of them:

$$\hat{f}(x, y) = \sum_{i=1}^{N} a_i h_i(x, y). \qquad (2)$$

Thus the null-space components of $\hat{f}$ are zero, which is a necessary condition for the minimum-norm solution. This leads to artifacts in $\hat{f}$ because it does not possess those components of f that lie in the null space. Further reading on the null-space–measurement-space concept may be found in papers by Twomey.[4–6]

The response-function expansion [Eq. (2)] is formally identical to the familiar backprojection process in which the value a_i is added to the image along the strip function h_i. Thus the backprojection process affects only the measurement-space components of the reconstruction. Most of the well-known CT reconstruction algorithms incorporate

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backprojection, including filtered backprojection,[7] the algebraic reconstruction technique (ART),[8] the simultaneous iterative construction technique (SIRT),[9] SIRT-like algorithms (least-squares[10] and other variants[11]), and the natural pixel matrix formulation by Buonocore *et al.*[12,13] Such algorithms can alter only the measurement-space part of the initial estimate. When the initial estimate lies solely in the measurement space, as is normally the case, so will the final estimate.

Various augmentations to deterministic algorithms, such as consistency, analytic continuation, and global constraints (including maximum entropy), have been considered by Hanson.[1] These seem to be ineffective in overcoming the measurement-space restrictions presented above for the solution of the general problem. Other authors have mentioned in passing the concept of the measurement-space–null-space dichotomy[14,15] but have not explicitly considered its effect on reconstructions from limited-projection data. As an aside, the range of the transpose of the projection-measurement matrix A, referred to in Ref. 15, is the measurement space in the square pixel representation. Smith *et al.*[16] considered the null space of a finite number of projections to explore the convergence rate of the ART algorithm. This work was extended by Hamaker and Solmon.[17] Katz made extensive use of the null-space concept to determine the conditions for uniqueness of a reconstruction.[18] Louis[19] developed an explicit expression for functions belonging to the null space corresponding to a finite set of projection data and showed that ghosts from the null space can appear as lesions. Medoff *et al.*[20] recognized the consequences of the null space associated with limited data and introduced a method to eliminate null-space ghosts through the application of known constraints on the reconstructed image. Further references on the limited-angle CT problem may be found in Ref. 1.

The restriction of deterministic solutions to the measurement space should not be viewed as a negative conclusion. Rather, it is simply a statement of what is possible for a given set of measurements in the absence of further information. It allows one to state formally the goal in an improved limited-angle CT reconstruction as that of estimating the null-space contribution through the use of further information about the function to be reconstructed.

BAYESIAN SOLUTION

The Bayesian approach to CT reconstruction[21] is based on the assumption that the image to be reconstructed belongs to an identifiable ensemble of similar images. In the following discussion, the image f and the set of all projections g are considered to be vectors. The best estimate for the reconstruction is taken to be that particular image f that maximizes the *a posteriori* conditional probability density of f given the measurements g. This probability is given by Bayes's law,

$$P(f|g) = \frac{P(g|f)P(f)}{P(g)}, \tag{3}$$

in terms of the conditional probability of g given f and the *a priori* probability distributions of f and g separately. We assume that the measurement noise is additive with a probability distribution that has zero mean and is Gaussian distributed. $P(f)$ is assumed to be a Gaussian distribution with a mean value $\bar{f}$. The covariance matrices of the noise and the ensemble image vectors are R_n and R_f, respectively. Under these assumptions, the MAP solution is easily shown to satisfy[22]

$$R_f^{-1}(\bar{f} - f) + H^T R_n^{-1}(g - Hf) = 0, \tag{4}$$

where H is the linear operator (matrix) corresponding to the projection process described by the integral in Eq. (1). The transpose of H is the familiar backprojection operation. It can be seen from Eq. (4) that the desired solution strikes a balance between its difference with the ensemble mean $\bar{f}$ and the solution to the measurement equation ($g = Hf$). This balance is determined by the covariance matrices R_f and R_n that specify the confidence with which each difference is weighted as well as possible correlations between the differences.

We have adopted an iterative approach to the solution of Eq. (4) based on the scheme proposed by Herman and Lent.[21] The nth estimate of f^n is given by the iteration scheme:

$$f^o = \bar{f}, \tag{5a}$$

$$f^{n+1} = f^n + c^n r^n, \tag{5b}$$

$$r^n = \bar{f} - f^n + R_f H^T R_n^{-1}(g - Hf^n), \tag{5c}$$

$$c^n = \frac{r^{nT} s^n}{s^{nT} s^n}, \tag{5d}$$

$$s^n = (I + R_f H^T R_n^{-1} H) r^n, \tag{5e}$$

where vector r^n is the residual of Eq. (4) (multiplied by R_f) and the scalar c^n is chosen to minimize the norm of r^{n+1}. This iterative scheme is similar to the one proposed by Hunt[22] for nonlinear MAP-image restoration. We have found that this technique works well, although convergence typically requires 10–20 iterations.

It is easy to see from the form of this iterative procedure that significant null-space contributions to f can arise from the *a priori* information. First, the zero-order estimate is $\bar{f}$, which can contain null-space contributions. Second, in Eq. (5c), R_f can generate null-space contributions when it operates on the results of the backprojection (H^T) process, which lies wholly in the measurement space. R_f, in effect, weights the backprojection of the measurement residuals. If R_f is chosen as zero in certain regions of the reconstruction, these regions will not be changed throughout the iteration scheme. It must be emphasized that the choices for $\bar{f}$ and R_f are exceedingly important since it is only through them that a nonzero null-space contribution to the reconstruction arises. As was stated earlier, this is the major advantage of the Bayesian approach over deterministic algorithms. Trivial choices, such as using for $\bar{f}$ a constant or a filtered backprojection reconstruction based on the same projections or assuming R_f to be proportional to the identity matrix,[22,23] are not helpful for reducing artifacts.

The iteration scheme given by Eqs. (5) is SIRT-like[11] in that the reconstruction update [Eq. (5b)] is accomplished only after all the ray sums (Hf^n) have been calculated. It is known that ART-like algorithms converge much faster than SIRT-like ones.[11] Herman *et al.*[23] have proposed an ART-like reconstruction algorithm that converges to the solution of the MAP equation [Eq. (4)] under the assumption that R_f and R_n are proportional to the identity matrix. This algorithm is worth exploring as it is likely to converge much more rapidly than the one used here. However, as was stated above, their algorithm should be extended to include nontrivial choices for R_f. The iterative scheme used here, although it may be slower than necessary, does provide a solution to the MAP equation, which is the important thing. We have found its convergence to be such that the norm of the residual [Eq. (5c)] behaves as the iteration number raised to the −1.5 to −2.0 power.

It is well known that the assumption of normal probability density distributions leads to a MAP solution [Eq. (4)] that is equivalent to the minimum-variance linear estimator.[24] Application of this estimator in a matrix formalism to tomographic reconstruction has been pursued by Wood *et al.*[25,26] and Buonocore *et al.*[27] These authors stressed the importance of *a priori* information in limited-angle reconstruction. However, the main thrust of their work was toward improving the computational efficiency of the required matrix operations, to which end they were successful.

Tam *et al.*[28] introduced a method to use the *a priori* known region of support of the source distribution. This method is the 2-D counterpart of the celebrated Gerchberg-Papoulis algorithm for obtaining superresolution. It is an iterative technique in which the known properties of the image in the spatial and the Fourier domains are alternately invoked. The objective is to use the known spatial extent of source image to extend its 2-D Fourier transform from the known sector into the missing sector. Although this method has been studied extensively[29–32] and has been shown to have some merit when either the region of support is restrictive or the angular region of the missing projections is fairly narrow, the region-of-support constraint may be incorporated more directly into many reconstruction algorithms. In virtually every iterative algorithm it is possible to invoke constraints on the reconstructed function. Thus it is possible to require the function to be zero outside the region of support at each update step, for example, in computing Eq. (5b) in the present MAP algorithm. Such an iterative algorithm will yield a solution that satisfies the region-of-support constraint, and Tam's procedure is not required. One may enforce this constraint through a redefinition of the response functions h_i in Eq. (1) to make them zero outside the region of support. Then the backprojection operation [Eq. (2)] affects only the reconstruction within the region of support. With this redefinition of h_i, the measurement space includes only functions that fulfill the region-of-support constraint. From this standpoint, Tam's iterative procedure does not affect the null space associated with the available measurements. The natural pixel formulation of Buonocore *et al.*[12,13] may be revised in a similar manner, but that would probably ruin the properties of the measurement matrix that he exploited to provide efficient matrix calculations.

FIT AND ITERATIVE RECONSTRUCTION

The preceding MAP solution is compared in the section on Results, below, with an alternative method to incorporate *a priori* information in the reconstruction process, namely, the FAIR technique introduced by Hanson.[3] In this algorithm the *a priori* information about the source function is used to construct a parameterized model of the unknown function. As the first step in this algorithm, the parameters in the model are determined from the available projection data by a least-squares (or minimum chi-squared) fitting procedure. In the second step of FAIR an iterative reconstruction procedure is performed using the fitted parametric model as the initial estimate. Although, in the past, the ART algorithm[8] was used in the second iterative step of FAIR, other iterative algorithms, such as the MAP algorithm above, can be used advantageously (see the Discussion section below). The iterative reconstruction procedure forces the result to agree with the measurements through alteration of its measurement-space contribution. The null-space contribution to the FAIR reconstruction arises solely from the parametric model fitted in the first step and hence from the *a priori* information used in specifying the model.

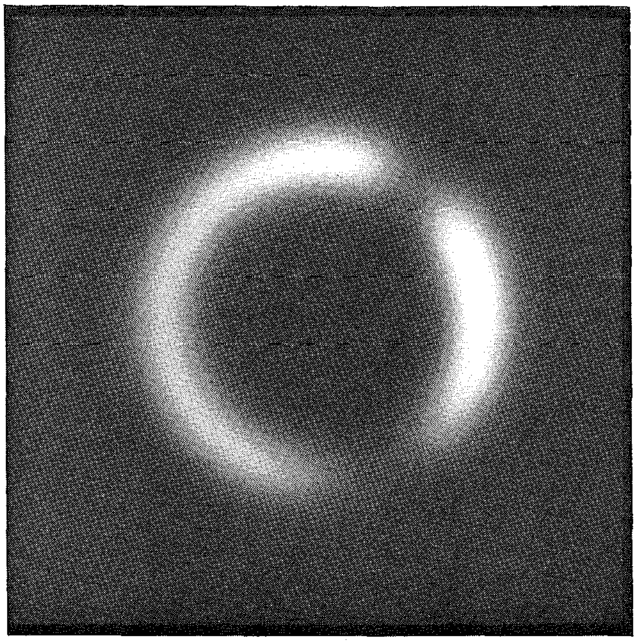
RESULTS

The results of application of various reconstruction methods to a specific 2-D, limited-angle reconstruction problem are compared. Algorithms that are useful for handling incomplete data through the use of *a priori* information must possess the following important characteristics: (1) significantly reduce artifacts that arise from inappropriate null-space contributions, (2) gracefully respond to inconsistencies between the actual source function and the assumptions about it, and (3) tolerate noise in the projection data. We demonstrate that the MAP and the FAIR algorithms conform to these requirements.

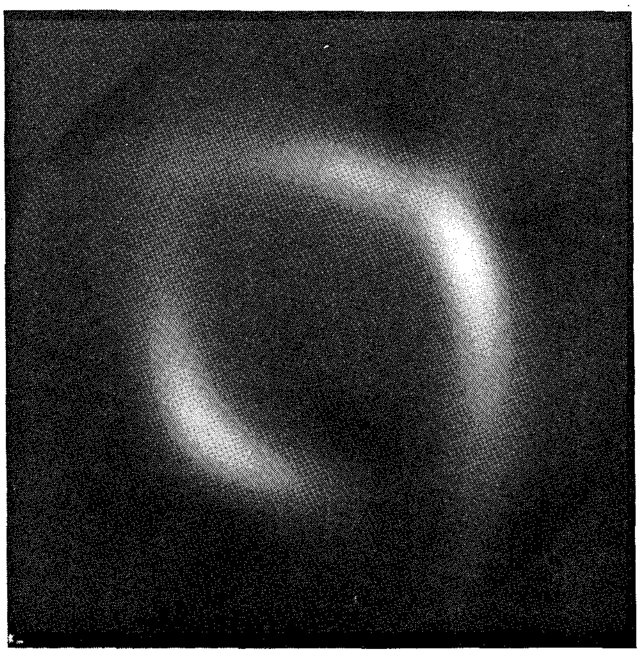
The relevant reconstruction techniques have been applied to an example source function consisting of a fuzzy annulus with variable amplitude [Fig. 1(a)], which roughly emulates the nuclear-isotope distribution in the cross section of a heart. The peak value of the distribution is 1.24. The available projection data consist of 11 views covering 90° in projection angle. At first no noise was added to the projections. Each projection contained 128 samples. All reconstructions contain 128 × 128 pixels. The measurement-space reconstruction obtained using ART[8] [Fig. 1(b)] shows severe artifacts that tend to obscure much of the source distribution.

Figure 1(c) shows the reconstruction obtained by using the maximum entropy algorithm (MENT) provided to us by Minerbo.[33] This algorithm provides a modest improvement over ART, particularly in regard to the detection of the dip in the annulus at 50°. However, MENT does not have much effect on the splaying of the reconstruction along the central axis of the available views. In our experience the principal advantage of the maximum-entropy constraint is its implicit constraint of nonnegativity. ART reconstructions that are constrained to be nonnegative are similar to the MENT results. The nonnegativity constraint amounts to the incorporation of *a priori* knowledge about the source function. This constraint is generally applicable and is effective in the reconstruction of certain types of source distributions, such as pointlike objects on a zero background. However, there are many source distributions and data-collection geometries for which nonnegativity provides little benefit, such as the present test case. We do not apply the nonnegativity constraint to any other reconstructions here, simply to avoid confusion about its role in providing improvement in the reconstructions as opposed to the role of the use of other *a priori* knowledge.

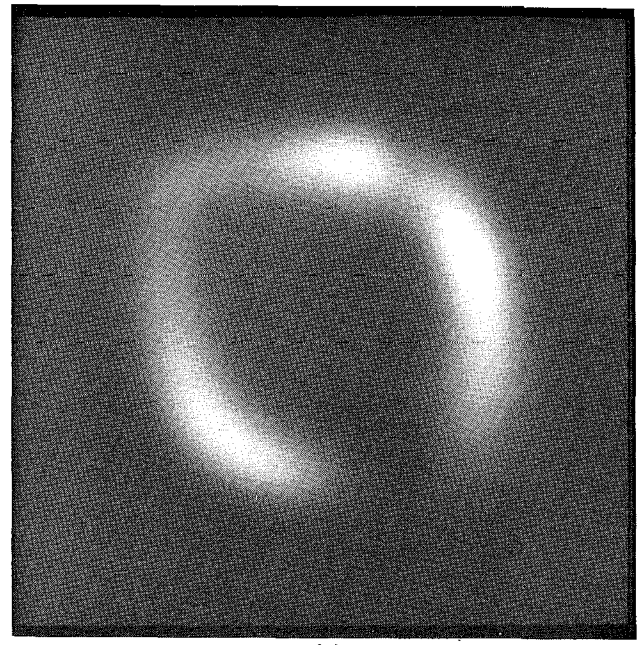
It was assumed that the *a priori* information consisted of the knowledge that the source function had an annular structure with known position, radius, and width. Thus, in



(a)



(b)



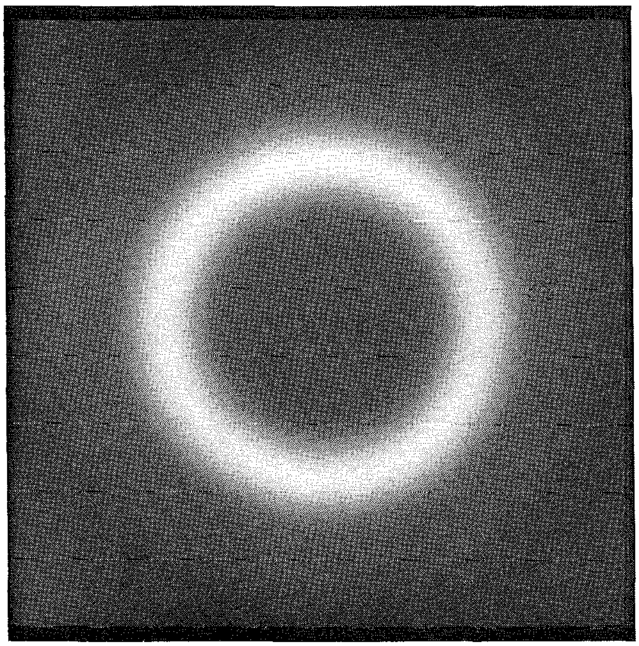
(c)

Fig. 1. (a) Source distribution used for the first example. (b) ART reconstruction and (c) MENT reconstruction obtained using 11 views covering 90° in projection angle. Unconstrained ART was used, whereas MENT has an implicit nonnegativity constraint.

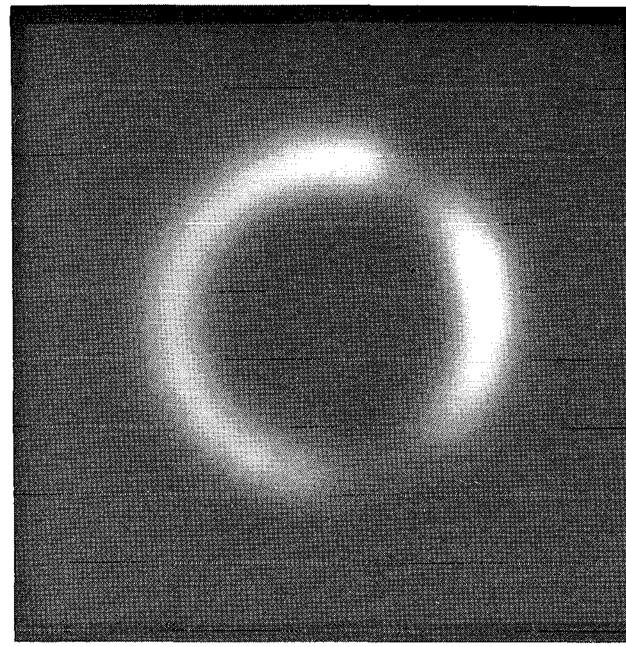
the MAP approach, $\bar{f}$ was chosen to be an annulus with constant amplitude and with Gaussian cross section. The mean radius and width of the annulus were chosen to be the same as the unknown source function. The covariance matrix R_f was assumed to be diagonal and was hence an image proportional to the ensemble variance about the mean $\bar{f}$. The covariance image R_f was large (1.0) at the peak of the annulus and small (0.2) inside and outside [Fig. 2(a)]. Since noiseless projections were used, the measurement noise was assumed to be uncorrelated, constant, and low in value. The resulting MAP reconstruction [Fig. 2(b)] is vastly superior to the ART and MENT results, eliminating essentially all the artifacts present in these deterministic solutions. The parametric model chosen for the FAIR method consisted of 18 2-D Gaussian functions evenly distributed on a circle. The radius of the circle and the width of the Gaussians were chosen to be the same as those of the source function. The fitting procedure determined the amplitudes of each of the Gaussian functions. The resulting fitted function was used as the initial estimate in ART to obtain the final result [Fig. 2(c)]. This FAIR reconstruction is comparable with the MAP result.

For a quantitative comparison, Fig. 3 shows the maximum reconstruction value obtained along radii as a function of angle for the various reconstruction methods presented in Figs. 1 and 2. The FAIR method is seen to follow the original source dependence most closely, with the MAP result a close second. The ART reconstruction has many quantitatively serious defects. The computation times on a CDC 7600 computer for the algorithms presented here are ART (10 iterations), 17 sec; MAP (10 iterations), 73 sec; FAIR (3 iterations), 25 sec; MENT (6 iterations), 105 sec. The corresponding execution time for filtered backprojection is 5 sec.

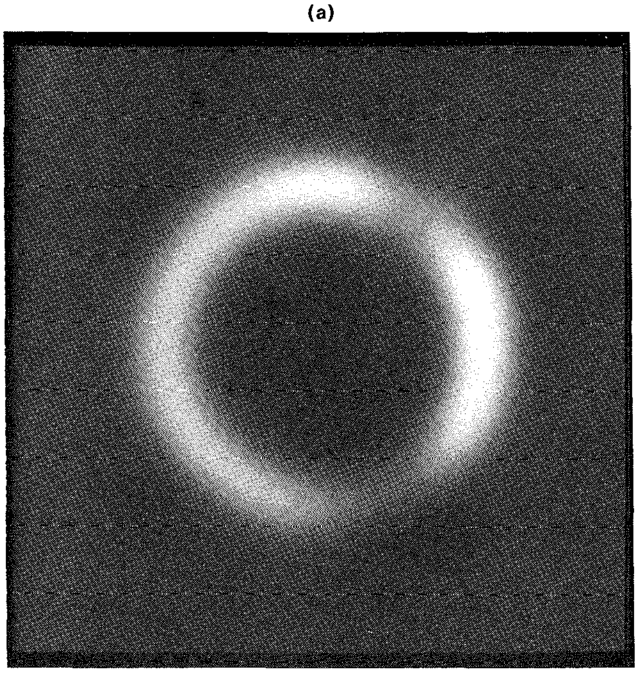
A slightly different source function, Fig. 4(a), was used to test the ability of the algorithms to deal with inconsistencies between assumptions about the source function and its actual distribution. This source function is the same as Fig. 1(a) with a narrow, 0.6-amplitude, 2-D Gaussian added outside the annulus at 330° and a broad, 0.1-amplitude Gaussian added underneath the annulus at 162°. The reconstructions obtained using the same assumptions as above are shown in Figs.



(a)



(c)



(b)

Fig. 2. Reconstructions using the *a priori* information that the unknown source function is a fuzzy annulus with known radius and width. (b) MAP reconstruction was obtained using a flat annulus for $\bar{f}$ and the variance image (a) as the diagonal entries of R_f (nondiagonal entries assumed to be zero). (c) The FAIR result was based on a model of the image consisting of 18 Gaussian functions distributed around the circle. The use of *a priori* knowledge significantly reduces the artifacts present in the deterministic reconstructioning in Fig. 1.

4(b)–4(d). Both MAP and FAIR handle the inconsistencies similarly. The angular dependence of the maximum reconstruction value, Fig. 5, shows that both algorithms produce an excess near 330° since they have tried to shift the discrepant exterior source to the annulus, which is consistent with the *a priori* assumptions. However, both methods do have a significant response in the region of the exterior source and therefore provide some information about the discrepancy. This would not be the case for the MAP algorithm were R_f chosen to be zero outside the annulus. This points out the need to be conservative in placing restrictions on the reconstruction that may be violated by the actual source distribution. The second iterative reconstruction step in the FAIR method is needed for this same reason as it allows correction

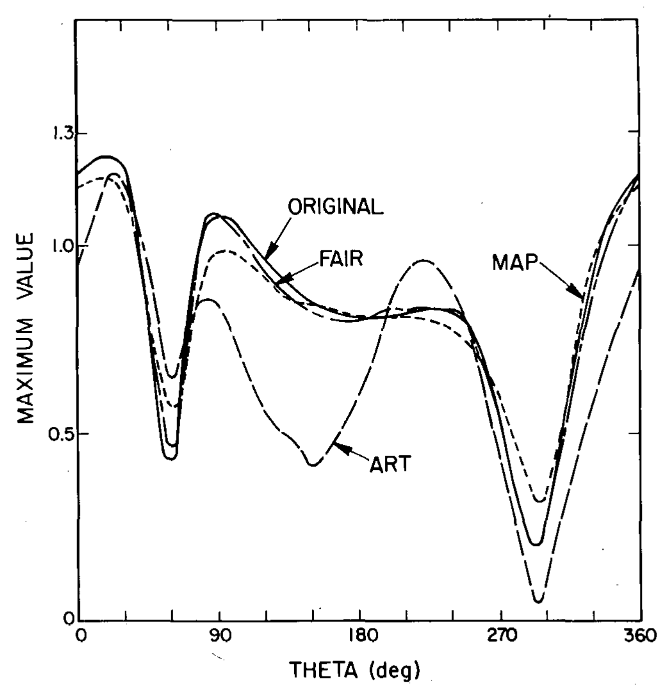


Fig. 3. The angular dependence of the maximum values along various radii for the ART, MAP, and FAIR reconstructions in Figs. 1 and 2 compared with that for the original function, Fig. 1(a), quantitatively demonstrates the improvement afforded by MAP and FAIR.

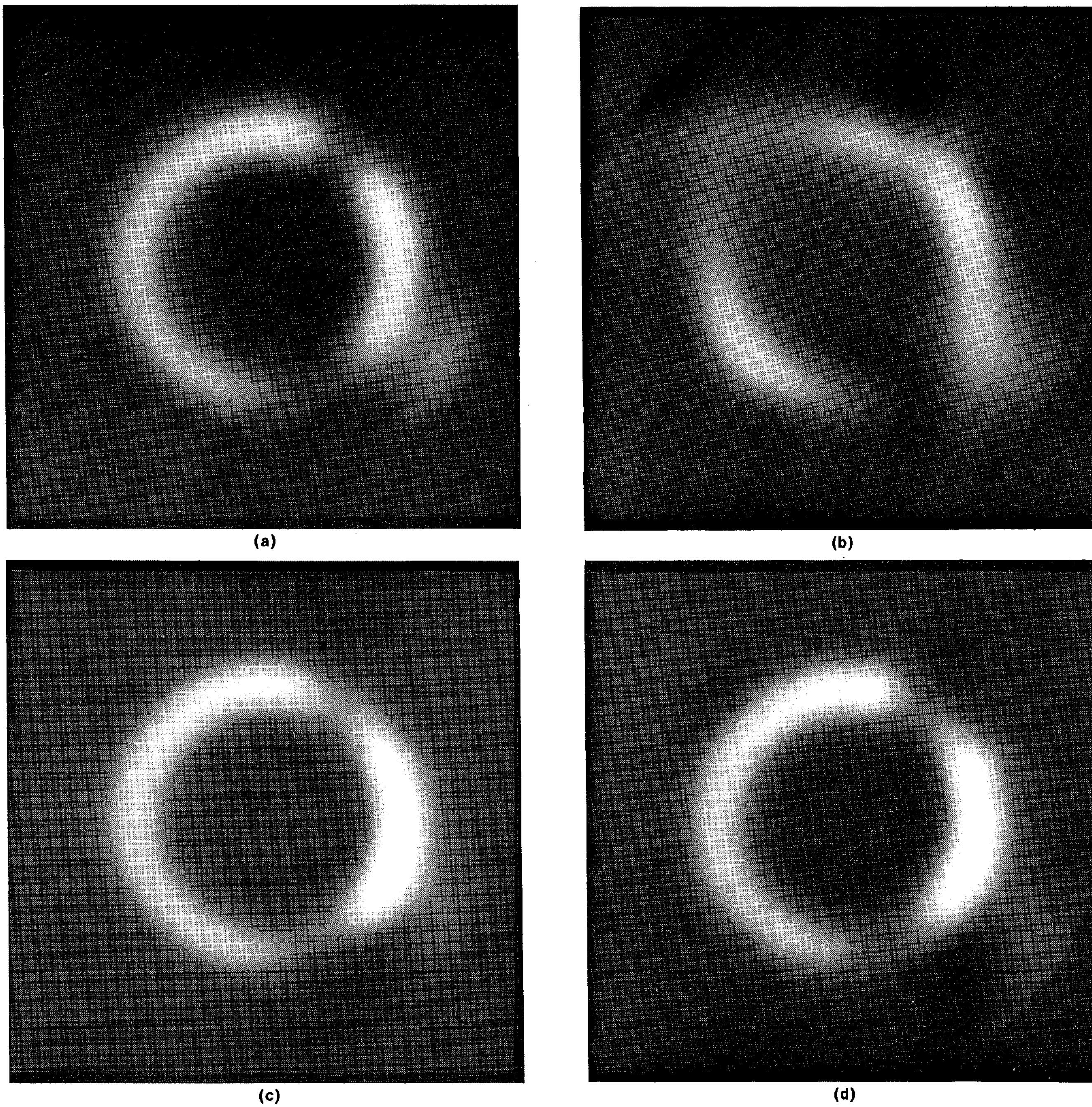


Fig. 4. Reconstructions of (a) source distribution that does not conform to the annular assumption obtained from 11 views subtending 90° using (b) ART, (c) MAP, and (d) FAIR algorithms. Both MAP and FAIR tend to move the added source outside the annulus onto the annulus. However, they provide indications in the reconstructions that there is some exterior activity.

to be made to the fitted model, if indicated by the available projections.

The final example is the reconstruction of Fig. 1(a) from noisy data. The same 11 projections were used as before but with random noise added with a rms deviation of 10% relative to the maximum projection value. The reconstructions in Fig. 6 demonstrate that both MAP and FAIR simply yield noisy versions of those obtained from noiseless projections. There is no disastrous degradation, as would be expected for algorithms based on analytic continuation.[34,35] Although the FAIR result appears to be much noisier than the MAP reconstruction, they both possess nearly identical noise in the annular region. The rms difference between the projection measurements and the ray sums of the MAP and the FAIR reconstructions, respectively, are roughly 0.8 and 0.5 times the actual rms deviation of the noise in the projections. This indicates that both algorithms have attempted to solve the measurement equations beyond what is reasonable. The

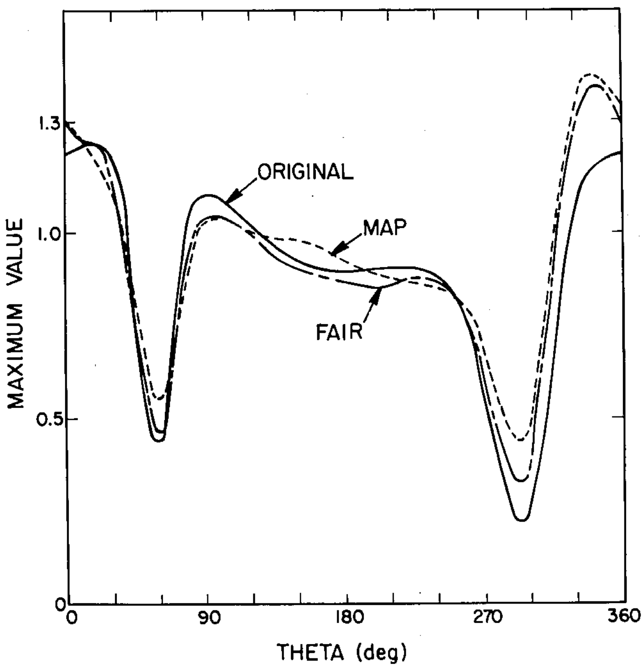


Fig. 5. Angular dependence of the maximum values in the MAP and the FAIR reconstructions of Fig. 4.

MAP algorithm does balance the rms error in the projections against the deviation from $\bar{f}$. However, ART simply attempts to reduce the rms projection error to zero.

DISCUSSION

In past comparisons of MAP results to more-standard techniques in the areas of CT[21,23] and image restoration,[36–38] the MAP approaches yielded little or no benefits. The reasons for the success of the MAP approach in the above limited-angle CT problem are: (1) The solution is severely underdetermined because of the limited data set. (2) The *a priori*

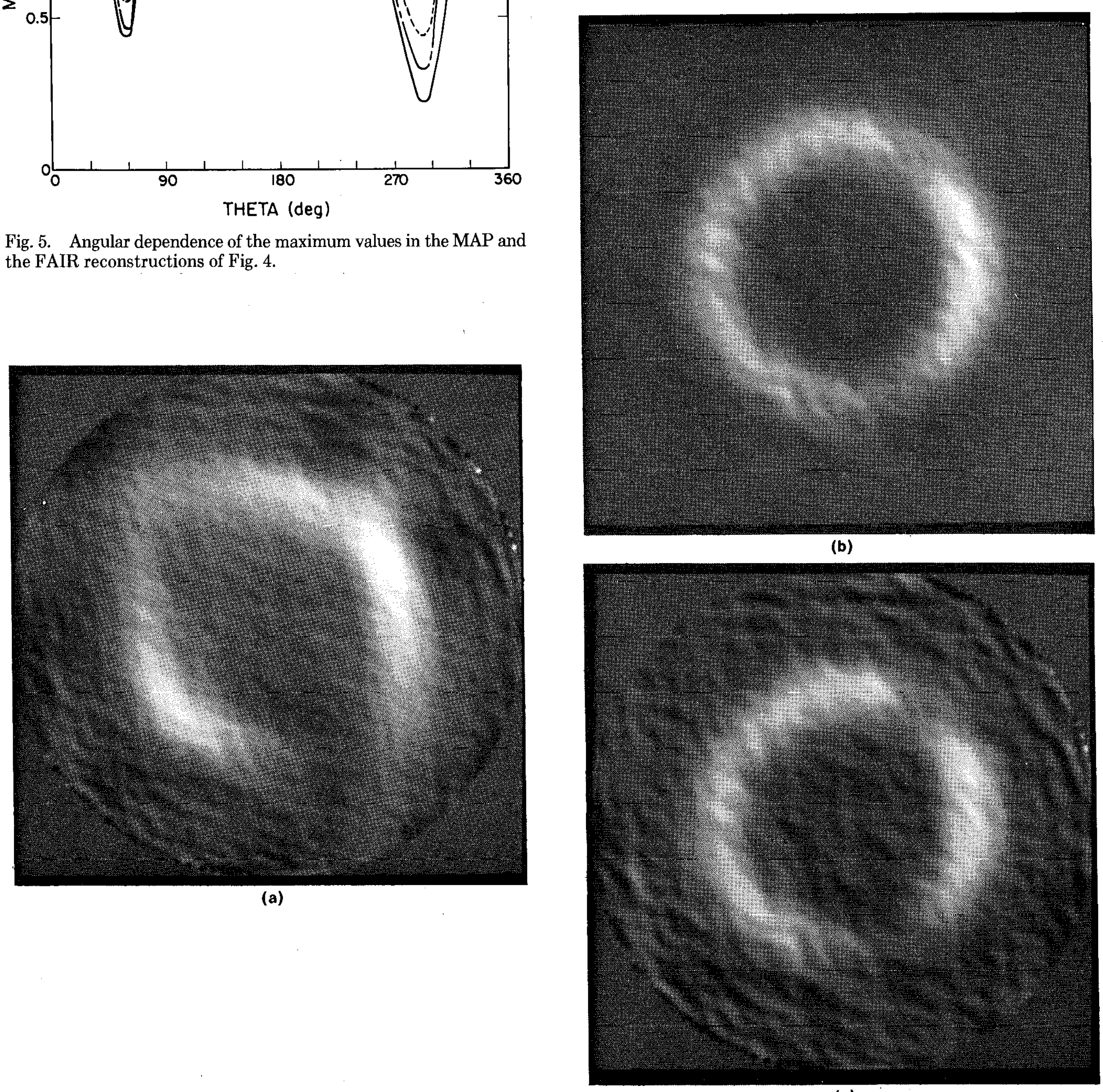


Fig. 6. Reconstructions of the source in Fig. 1(a) from 11 noisy projections using (a) ART, (b) MAP, and (c) FAIR algorithms show that the latter two algorithms are tolerant of noise.

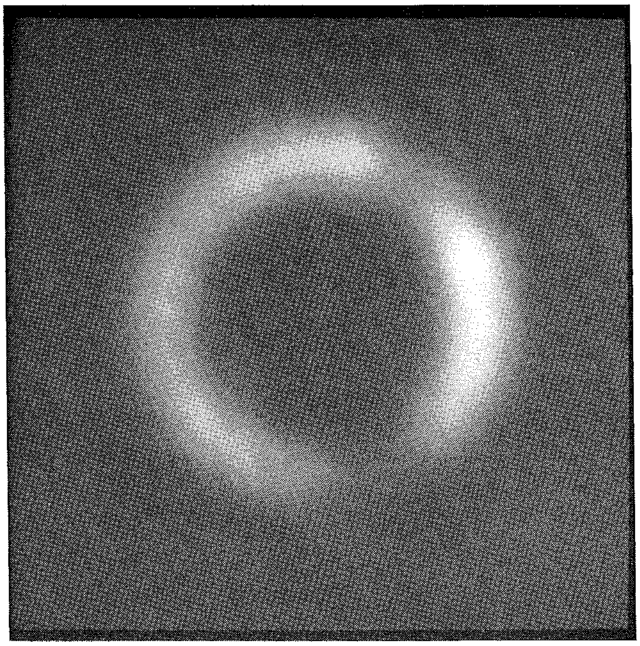


Fig. 7. Reconstruction from the same data as used in Fig. 6 obtained by employing MAP as the second step in the FAIR procedure. This global Bayesian approach yields the best estimate of the original function and provides flexibility in the use of *a priori* information.

assumptions about $\bar{f}$ and R_f can be made quite restrictive. It is expected that the MAP analysis will be most useful in situations in which these two conditions hold.

The incorporation of *a priori* knowledge in the MAP algorithm presented above is quite restricted. It does not readily accommodate source distributions that vary in size, shape, or location. However, the fitting procedure used in the first step of FAIR can easily handle such variations by including them as variables to be determined from the data. In the spirit of the Bayesian approach, constraints on these variables may be introduced to guide the fitting procedure toward a reasonable result. The use of ART in the second iterative portion of FAIR has the disadvantage that ART tries to reduce the discrepancy in the measurement equations to zero without regard for the estimated uncertainties in the data. Thus the FAIR result shown in Fig. 6(c) is quite noisy and is substantially further from the actual source distribution (rms deviation = 0.154) than the intermediate fitted result (rms deviation = 0.031).

In a more-global Bayesian approach to the problem, the fitting procedure in FAIR may be used to estimate $\bar{f}$ and R_f for input to a MAP algorithm. The fitting procedure may be viewed as defining a subensemble appropriate to the available data. For the present example, the fitted result was used for $\bar{f}$, and R_f was assumed to be 0.1 times the identity matrix. R_f is assumed to be much smaller than that used in the preceding MAP calculation to reflect the supposition that the fit is much closer to the desired result than is the annulus of constant amplitude used for $\bar{f}$ previously. The resulting MAP reconstruction, using the 10%-rms noise data, shown in Fig. 7 is better than any of the results shown in Fig. 6. The rms deviation of this reconstruction relative to the source function is 0.035, whereas that for the earlier MAP result, Fig. 6, is 0.060. When this FAIR–MAP method is applied to the projections of the inconsistent source function shown in Fig. 4(a), the result is similar to that obtained with FAIR using ART [Fig. 4(c)]. These examples demonstrate the power of this global Bayesian approach in which the MAP algorithm is used for the second, iterative step of FAIR.

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